H.W. 9 (solutions)

1 $S = 1 + 7 + 13 \cdots\cdots + 601$

$601 = 1 + 6n$

$\boxed{n = 100}$

Hence S has 101 terms.

$S = 1 + 7 + 13 + \cdots\cdots + 601$

$= 601 + \cdots\cdots\cdots\cdots + 1$

$2S = \underbrace{602 + 602 + \cdots\cdots + 602}_{101 \text{ terms}}$

$= \cancel{10} \; 101 \times 602$.

$\boxed{S = 101 \times 301}$

2

$$S = 6 + 6(.1) + 6(.1)^2 + \cdots + 6(.1)^N.$$

$$.1S = 6(.1) + 6(.1)^2 + \cdots + 6(.1)^N + 6(.1)^{N+1}$$

$$S - .1S = 6 - 6(.1)^{N+1}$$

$$(1 - .1)S = 6\left[1 - (.1)^{N+1}\right]$$

$$S = \frac{6\left[1 - (.1)^{N+1}\right]}{.9}$$

$$= \frac{60}{9}\left[1 - (.1)^{N+1}\right]$$

$$S_{10} = \frac{60}{9}\left[1 - (.1)^{11}\right]$$

$$S_{100} = \frac{60}{9}\left[1 - (.1)^{101}\right].$$

③

③ $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges because

$$\int_1^{\infty} \frac{1}{x^5}\,dx = \left.\frac{x^{-4}}{-4}\right|_1^{\infty}$$

$$= \underset{N\to\infty}{Lt}\ \cancel{z}\ \left.-\frac{1}{4x^4}\right|_1^{N}$$

$$= \underset{N\to\infty}{Lt} -\frac{1}{4}\left[\frac{1}{N^4} - 1\right]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{4}\left[1 - \frac{1}{N^4}\right] = \frac{1}{4}.$$

4

4 $x_n = \left(.5 + \frac{1}{n}\right)^n$

$x_{n+1} = \left(.5 + \frac{1}{n+1}\right)^{n+1}$

ratio = $\dfrac{\left(.5 + \frac{1}{n+1}\right)^{n+1}}{\left(.5 + \frac{1}{n}\right)^n}$

write $.5 + \frac{1}{n} = .5\left[1 + \frac{2}{n}\right]$

$.5 + \frac{1}{n+1} = .5\left[1 + \frac{2}{n+1}\right]$

ratio = $\dfrac{.5^{n+1}\left[1 + \frac{2}{n+1}\right]^{n+1}}{.5^n\left[1 + \frac{2}{n}\right]^n}$

$= .5 \dfrac{\left(1 + \frac{2}{n+1}\right)^{n+1}}{\left(1 + \frac{2}{n}\right)^n}$

$$\underset{n\to\infty}{Lt} \text{ ratio} = \cdot 5 \frac{\underset{n\to\infty}{Lt}\left(1+\frac{2}{n+1}\right)^{n+1}}{\underset{n\to\infty}{Lt}\left(1+\frac{2}{n}\right)^{n}}$$

$$= \cdot 5 \; \frac{e^2}{e^2} = \cdot 5 .$$

$\rho = \cdot 5$

Hence the series converges.

⑥

⑤

$$\int_1^{\infty} \frac{2x}{1+x^2}\,dx = \int_2^{\infty} \frac{du}{u} = \underset{a\to\infty}{Lt}\ \ln u \Big|_2^a$$

$1+x^2=u$

$du = 2x\,dx$

$$\underset{a\to\infty}{Lt}\ \ln a - \ln 2$$

$$= \underset{a\to\infty}{Lt}\ \ln a - \ln 2$$

Diverges.

Hence the sequences diverges.

~~6~~ 5 (b)

$$\int_1^{\infty} \frac{2x}{(1+x^2)^2}\,dx = \int_2^{\infty} \frac{du}{u^2}$$

$$\underset{a\to\infty}{Lt} - \frac{1}{u}\Big|_2^a$$

$$\underset{a\to\infty}{Lt} - \frac{1}{a} + \frac{1}{2} = \frac{1}{2}.$$

$\boxed{\text{Converges}}$

7

6

$$\frac{1}{1-x}$$

Look at

$S = 1 + x + x^2 + \cdots + x^n$

$xS = x + x^2 + \cdots + x^{n+1}$

$S - xS = 1 - x^{n+1}$

$\Rightarrow S = \dfrac{1-x^{n+1}}{1-x}$

If $|x|<1$ $\quad S = \dfrac{1}{1-x}$. when $n \to \infty$

Hence

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

8

(b) write

$$y = \frac{1}{x}$$

so that when

$|x| > 1$ we have $|y| < 1$

$$\frac{1}{1-x} = \frac{1}{1-\frac{1}{y}} = \frac{y}{y-1}$$

$$= -y\left(\frac{1}{1-y}\right)$$

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots \quad \text{when } |y| < 1$$

$$= 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots \quad \text{when } |x| > 1.$$

$$\therefore \frac{1}{1-x} = -\frac{1}{x}\left[1 + \frac{1}{x} + \frac{1}{x^2} + \cdots\right]$$

$$= -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots$$

when $|x| > 1$.

7

9

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + \cdots\cdots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots}$$

— x —

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f''''(x) = \sin x \qquad f''''(0) = 0$$

$$\vdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\cdots$$

$$f(x) = \cos x \quad f(0) = 1$$

$$f'(x) = -\sin x.$$

$$f''(x) = -\cos x \quad f''(0) = -1.$$

$$f'''(x) = +\sin x.$$

$$f''''(x) = \cos x \quad f''''(0) = 1.$$

$$\vdots \qquad \vdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$$