H. W. 8

Solutions

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(i) $$\int_0^\infty \frac{dx}{x^2+1}$$

$$\int_0^N \frac{dx}{x^2+1} = \tan^{-1}x \Big|_0^N$$

$$= \tan^{-1}N - \tan^{-1}0$$

$$\int_0^\infty \frac{dx}{x^2+1} = \underbrace{\operatorname{Lt}_{N\to\infty} \tan^{-1}N - \tan^{-1}0}$$

Restricting $\tan^{-1}$ to have a range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have

$$\operatorname{Lt}_{N\to\infty} \tan^{-1}N = \pi/2$$

$$\tan^{-1}0 = 0$$

$$\therefore \int_0^\infty \frac{dx}{x^2+1} = \pi/2$$

(ii) $\displaystyle\int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$

$$\int_N^1 x^{-1/2}\,dx = 2\sqrt{x}\Big|_N^1$$

$$= 2\left[\sqrt{1} - \sqrt{N}\right]$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \underset{N \to 0+}{Lt}\ 2\left[\sqrt{1} - \sqrt{N}\right]$$

$$= 2$$

(iii) $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ $\qquad$ $y = 4 - x$

$\qquad\qquad\qquad\qquad$ $dy = -dx$

$$= \int_4^0 \frac{-dy}{\sqrt{y}} = \int_0^4 \frac{dy}{\sqrt{y}} = 2\left[\sqrt{4}\right]$$

$$= 4$$

(iv) $\int_0^\infty e^{-x}\cos x \, dx$

$$\int e^{-x}\cos x \, dx$$

$$= e^{-x}\left[\frac{\sin x - \cos x}{2}\right] + c$$

$$\int_0^N e^{-x}\cos x \, dx$$

$$= e^{-N}\left[\frac{\sin N - \cos N}{2}\right] - \left[-\frac{1}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{2} e^{-N}(\sin N - \cos N)$$

$$\int_0^\infty e^{-x}\cos x \, dx = \frac{1}{2} + \frac{1}{2} \underset{N\to\infty}{Lt}\, \underbrace{e^{-N}(\sin N - \cos N)}_{\to 0}$$

$$= 1/2$$

2 (i)

$$\ln\frac{1}{2}+\ln\frac{2}{3}+\cdots+\ln\frac{n}{n+1}$$

$$\begin{aligned}
&= (\ln 1-\ln 2)\\
&+(\ln 2-\ln 3) \qquad = \ln 1-\ln(n+1)\\
&+ \cdots\cdots \qquad\qquad = -\ln(n+1)\\
&+ \cdots\cdots\\
&+(\ln n-\ln(n+1))
\end{aligned}$$

(ii) $S_n = 2\left[1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^{n-1}}\right]$

$$\frac{1}{3}S_n = 2\left[\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^{n-1}}+\frac{1}{3^n}\right]$$

$$S_n-\frac{1}{3}S_n = 2\left[1-\frac{1}{3^n}\right]$$

$$S_n = \frac{3}{\cancel{2}} \cdot \cancel{2}\left[1 - \frac{1}{3^n}\right]$$

$$= 3\left[1 - \frac{1}{3^n}\right]$$

3) Ans:

The series

$$1 + x + x^2 + \cdots$$

converges in $|x| < 1$. This follows from the ration test. Also if $|x| > 1$, the series diverges. Finally if $|x| = 1$ then $x = 1$ or $x = -1$.

When $x = 1$, the series diverges.

" $x = -1$, " " does not converge either.

(4)

The series

$$-\frac{1}{x}\left[1+\frac{1}{x^2}+\frac{1}{x^2}+\cdots\right]$$

converges if $\left|\frac{1}{x}\right|<1$

ie $|x|>1$.

Also follows from ratio test.

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(5) (i) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

This series converges iff

$\displaystyle\int_1^{\infty}\frac{x}{x^2+1}\,dx$ converges

If $y = x^2+1$

$dy = 2x\,dx$

$$\int \frac{x\,dx}{x^2+1} = \int \frac{dy}{2}\frac{1}{y}$$

$$= \frac{1}{2}\ln|y|$$

$$= \frac{1}{2}\ln|x^2+1|.$$

$$\int_1^N \frac{x\,dx}{x^2+1} = \frac{1}{2}\left[\ln(N^2+1) - \ln 2\right]$$

$$\int_1^\infty \frac{x\,dx}{x^2+1} = \frac{1}{2}\left[\underset{N\to\infty}{Lt}(N^2+1) - \ln 2\right]$$

The integral diverges and hence the sequence diverges as well ii

$$m = n - 2$$

$$\sum_{m=1}^{\infty} \frac{1}{(m+2)\ln(m+2)}$$

$$\int \frac{1}{(x+2)\ln(x+2)}\,dx$$

$$y = x + 2$$

$$= \int \frac{dy}{y \ln y}$$

$$\ln y = w$$

$$dw = \frac{1}{y}\,dy.$$

$$= \int \frac{dw}{w} = \ln|w|$$

$$= \ln|\ln y|$$

$$= \ln|\ln(x+2)|$$

$$\int_1^N \frac{1}{(x+2)\ln(x+2)}\,dx$$

$$= \ln|\ln(N+2)| - \ln[\ln 3]$$

$$\int_1^\infty \frac{1}{(x+2)\ln(x+2)}\,dx =$$

$$\lim_{N\to\infty} \ln[\ln(N+2)] - \ln(\ln 3)$$

The limit diverges.

Hence the sequence diverges.

(iii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$u_n = \frac{n!}{n^n}$$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!}$$

$$= \frac{(n+1)\ n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n}$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$\underset{n \to \infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n \to \infty}{Lt} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1$$

series converges

(iv)

$$u_n = \frac{(n+1)(n+2)}{n!}$$

$$u_{n+1} = \frac{(n+2)(n+3)}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+2)(n+3)}{(n+1)!} \frac{n!}{(n+1)(n+2)}$$

$$= \frac{\cancel{(n+2)}(n+3)}{(n+1)^2 \cancel{(n+2)}}$$

$$= \frac{n+3}{(n+1)^2}$$

$$\underset{n \to \infty}{Lt} \frac{u_{n+1}}{u_n} = 0 < 1$$

series converges.