Calculus

II

H. W. 6 Solutions.

1 Ans:

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

$$= \cos^{n-1} x \sin x - \int (n-1)\left[\cos^{n-2} x\right](-1)\sin x \cdot \sin x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \left[\cos^{n-2} x\right] \sin^2 x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \left(1-\cos^2 x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$- (n-1) \int \cos^n x \, dx$$

Hence

$$\int \cos^n x\,dx + (n-1)\int \cos^n x\,dx$$

$$= [\cos^{n-1} x][\sin x] + (n-1)\int \cos^{n-2} x\,dx$$

$$L.H.S = n\int \cos^n x\,dx.$$

$$\therefore \int \cos^n x\,dx =$$

$$\frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$\blacksquare$

2 Ans:

$$\int \sin^n x \, dx =$$

$$\int \sin^{n-1} x \sin x \, dx$$

$$= \sin^{n-1} x \cos x (-1) - \int (n-1) \sin^{n-2} x \cos x \cdot \cos x (-1) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$

$$- \cancel{\#} (n-1) \int \sin^n x \, dx.$$

Hence

$$\int \sin^n x\,dx + (n-1)\int \sin^n x\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$

$$L.H.S = n\int \sin^n x\,dx$$

Hence

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

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3 Ans:

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\int \tan^{-1} x \, dx =$$

$$\int \tan^{-1} x \cdot 1 \, dx$$

$$= [\tan^{-1} x][x] - \int \frac{1}{1+x^2} x \, dx$$

$$= x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$\int \frac{x}{1+x^2} dx$$

$u = 1 + x^2$

$du = 2x \, dx$

$$= \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| = \frac{1}{2} \ln|1+x^2|.$$

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$$\int \tan^{-1} x \, dx =$$

$$x \cdot \tan^{-1} x - \frac{1}{2} \ln |1 + x^2|$$

4 Ans:

$$I_1 = \int e^{ax} \cos bx \, dx$$

$$= e^{ax} \frac{\sin bx}{b} - \int a e^{ax} \frac{\sin bx}{b} dx$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx$$

$$= I_2$$

$$I_1 = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} I_2$$

$$I_2 = \int e^{ax} \sin bx \, dx$$

$$= e^{ax} \frac{\cos bx}{b}(-1)$$

$$- \int a e^{ax} \frac{\cos bx}{b}(-1) \, dx$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \underbrace{\int e^{ax} \cos bx \, dx}_{= I_1}$$

$$I_2 = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} I_1$$

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We need to simultaneously solve

$$I_1 = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} I_2 .$$

$$I_2 = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} I_1 .$$

$$\Downarrow$$

$$I_1 = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b}\left[-\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} I_1\right]$$

$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} I_1$$

$$\Rightarrow \left(1 + \frac{a^2}{b^2}\right) I_1 = \frac{1}{b} e^{ax}\left[\sin bx + \frac{a}{b} \cos bx\right]$$

$$\Rightarrow I_1 = \frac{b}{a^2+b^2} e^{ax}\left[\sin bx + \frac{a}{b} \cos bx\right]$$

$$\Rightarrow I_1 = e^{ax}\left[\frac{b\sin bx + a\cos bx}{a^2+b^2}\right]$$

— x —

$$I_2 = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}I_2\right]$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I_2.$$

$$= -\frac{1}{b}e^{ax}\left[\cos bx - \frac{a}{b}\sin bx\right] - \frac{a^2}{b^2}I_2.$$

$$\left(1+\frac{a^2}{b^2}\right) I_2 = -\frac{1}{b} e^{ax}\left[\frac{b\cos bx - a\sin bx}{b}\right]$$

$$\frac{a^2+b^2}{\cancel{b^2}} I_2 = -\frac{1}{\cancel{b^2}} e^{ax}\left[b\cos bx - a\sin bx\right]$$

$$I_2 = e^{ax}\left[\frac{a\sin bx - b\cos bx}{a^2+b^2}\right]$$

Summary:

$$\int e^{ax}\cos bx\,dx = e^{ax}\left[\frac{b\sin bx + a\cos bx}{a^2+b^2}\right]$$

$$\int e^{ax}\sin bx\,dx = e^{ax}\left[\frac{a\sin bx - b\cos bx}{a^2+b^2}\right]$$

5 Ans!

a $x^2-2x-3=(x+1)(x-3)$.

$$\therefore \frac{5x-3}{x^2-2x-3}=\frac{A}{x+1}+\frac{B}{x-3}.$$

$$A=\left.\frac{5x-3}{x-3}\right|_{x=-1}=\frac{-5-3}{-1-3}=\frac{8}{4}=2$$

$$B=\left.\frac{5x-3}{x+1}\right|_{x=3}=\frac{15-3}{3+1}=\frac{12}{4}=3.$$

$$\therefore \frac{5x-3}{x^2-2x-3}=\frac{2}{x+1}+\frac{3}{x-3}.$$

$$\int\frac{5x-3}{x^2-2x-3}dx=2\int\frac{dx}{x+1}+3\int\frac{dx}{x-3}.$$

$$=2\ln|x+1|+3\ln|x-3|.$$

b

$$\frac{4-2x}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}.$$

Multiplying by $(x-1)^2$ and set $x=1$ we get

$$\left.\frac{4-2x}{x^2+1}\right|_{x=1} = D$$

$$\therefore D = \frac{4-2}{1+1} = \frac{2}{2} = 1.$$

$$\therefore \frac{4-2x}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{1}{(x-1)^2}.$$

$$= \frac{(Ax+B)(x-1)^2 + C(x^2+1)(x-1) + (x^2+1)}{(x^2+1)(x-1)^2}.$$

$$(Ax+B)(x-1)^2 =$$

$$(Ax+B)(x^2-2x+1).$$

$$= Ax^3 - 2Ax^2 + Ax$$
$$+ Bx^2 - 2Bx + B$$

$$= Ax^3 + (B-2A)x^2 + (A-2B)x + B.$$

$$C(x^2+1)(x-1) = C\left[x^3 + x - x^2 - 1\right]$$

— x —

Thus:

$$(Ax+B)(x-1)^2 + C(x^2+1)(x-1) + x^2+1.$$
$$= 4-2x$$

$$\Rightarrow (A+C)x^3 + (B-2A-C+1)x^2.$$
$$+ (A-2B+C)x = -2x+4$$
$$+ (B-C+1)$$

Comparing co-efficients we obtain.

$$A + C = 0.$$
$$B - 2A - C + 1 = 0.$$
$$A - 2B + C = -2.$$
$$B - C + 1 = 4.$$

We have $A = -C$ & $B = C + 3$.
Plugging this into above eqn we get

$$A - 2B + C = -2$$
$$\Rightarrow -C - 2(C+3) + C = -2$$
$$\Rightarrow -C - 2C - 6 + C = -2$$
$$\Rightarrow -2C = 4 \Rightarrow C = -2.$$
$$\therefore A = 2,\ B = 1$$

$$\therefore \frac{4-2x}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

$$\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx =$$

$$\int \frac{2x+1}{x^2+1}\,dx - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$

$$= 2\int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$- 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$

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(i) $\int \frac{2x}{x^2+1}\,dx$ $\quad u = x^2+1$

$du = 2x\,dx$.

$= \int \frac{du}{u}$

$= \ln|u| = \ln|x^2+1|$.

(ii) $\int \frac{dx}{x^2+1} = \tan^{-1} x$.

(iii) $-2\int \frac{dx}{x-1} = -2\ln|x-1|$.

(iv) $\int \frac{dx}{(x-1)^2} = \frac{-1}{(x-1)}$

It follows that

$$\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx =$$

$$\ln|x^2+1| + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1}.$$

$$= \ln\left[\frac{x^2+1}{(x-1)^2}\right] - \frac{1}{x-1} + \tan^{-1}x.$$

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6 Ans:

$$\int \frac{dx}{(x^2+4)^2}$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$.

$$= \int \frac{2\sec^2\theta \, d\theta}{(4\tan^2\theta + 4)^2}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{16\sec^4\theta} = \frac{1}{8}\int \cos^2\theta \, d\theta$$

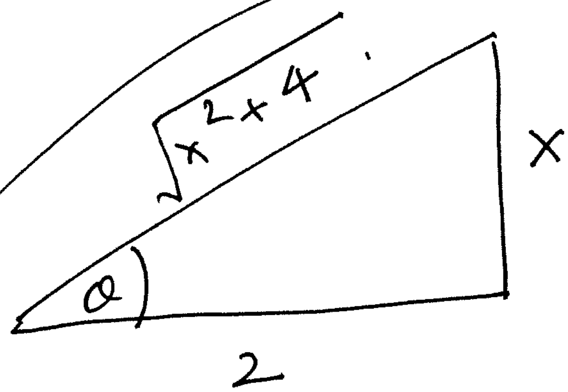
$$= \frac{1}{8}\int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{16}\int d\theta + \frac{1}{16}\int \cos 2\theta \, d\theta.$$

$$= \frac{1}{16}\theta + \frac{1}{16} \cdot \frac{\sin 2\theta}{2}.$$

$$= \frac{\theta}{16} + \frac{1}{32}\sin 2\theta.$$

$$= \frac{1}{16}\tan^{-1}\left[\frac{x}{2}\right] + \frac{1}{32}\sin 2\theta$$



$$\sin\theta = \frac{x}{\sqrt{x^2+4}}.$$

$$\cos\theta = \frac{2}{\sqrt{x^2+4}}.$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{4x}{x^2+4}.$$

$$= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{8}\,\frac{x}{x^2+4}. + C$$

7 **Ans:**

$$\frac{3x+2}{x^2+2x+3} = \frac{3x+2}{(x^2+2x+1)+2}$$

$$= \frac{3x+2}{(x+1)^2+2}.$$

$$= \frac{3(x+1)-1}{(x+1)^2+2}$$

$$= 3\,\frac{x+1}{(x+1)^2+2}. - \frac{1}{(x+1)^2+2}.$$

$$\int \frac{3x+2}{x^2+2x+3}\,dx = 3\int \frac{x+1}{(x+1)^2+2}\,dx.$$

$$- \int \frac{1}{(x+1)^2+2}\,dx.$$

Choose
$u = x+1$
$du = dx$

$$= 3\int \frac{u}{u^2+2}\,du - \int \frac{du}{u^2+2}.$$

$$3\int \frac{u\,du}{u^2+2} = \frac{3}{2}\int \frac{2u\,du}{u^2+2}$$

$v = u^2+2$
$dv = 2u\,du$

$$= \frac{3}{2}\int \frac{dv}{v}$$

$$= \frac{3}{2}\ln|v|$$

$$= \frac{3}{2}\ln|u^2+2|.$$

$$= \frac{3}{2}\ln|x^2+2x+1+2|.$$

$$= \frac{3}{2}\ln|x^2+2x+3|.$$

$$\int \frac{du}{u^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right).$$

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$$\int \frac{3x+2}{x^2+2x+3}\,dx =$$

$$\frac{3}{2}\ln\left|x^2+2x+3\right| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right).$$

7 b Ans:

$$\int \frac{3x+2}{(x^2+2x+5)^2}\,dx$$

$$= \int \frac{3x+2}{[(x+1)^2+4]^2}\,dx$$

$u = x+1$

$du = dx$

$x = u-1$

$$= \int \frac{[3(u-1)+2]\,du}{(u^2+4)^2}$$

$$= \int \frac{3u-1}{(u^2+4)^2}\,du = \frac{3}{2}\int \frac{2u}{(u^2+4)^2}\,du - \int \frac{du}{(u^2+4)^2}$$

$$\int \frac{2u}{(u^2+4)^2}\,du.$$

$$= \int \frac{dv}{v^2}$$

$v = u^2+4$
$dv = 2u\,du.$

$$= -\frac{1}{v} = -\frac{1}{u^2+4}.$$

$$\int \frac{du}{(u^2+4)^2} = \frac{1}{16}\tan^{-1}\frac{u}{2} + \frac{1}{8}\frac{u}{u^2+4}.$$

problem 6

$$\therefore \int \frac{3x+2}{(x^2+2x+5)^2}\,dx = -\frac{3}{2}\frac{1}{u^2+4}$$

$$-\frac{1}{16}\tan^{-1}\frac{u}{2}$$

$$-\frac{1}{8}\frac{u}{u^2+4}.$$

$$= -\frac{1}{8}\left[\frac{12+u}{u^2+4}\right] - \frac{1}{16}\tan^{-1}\frac{u}{2}.$$

$$= -\frac{1}{8}\left[\frac{x+13}{(x+1)^2+4}\right] - \frac{1}{16}\tan^{-1}\left[\frac{x+1}{2}\right]$$

8 Ans:—

a $\int \frac{\sin\theta \, d\theta}{\cos^2\theta + \cos\theta - 2}$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$= \int \frac{-du}{u^2+u-2} = \int \frac{-du}{\left(u+\frac{1}{2}\right)^2 - 2 - \frac{1}{4}}$

$2 + \frac{1}{4} = \frac{8+1}{4} = \frac{9}{4}$.

$= \int \frac{-du}{\left(u+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$

$v = u + \frac{1}{2}$

$dv = du$

$$= \int \frac{-\,dv}{v^2 - \left(\frac{3}{2}\right)^2} = \int \frac{dv}{\left(\frac{3}{2}\right)^2 - v^2}$$

write

$v = \frac{3}{2}\sin\alpha$

$dv = \frac{3}{2}\cos\alpha\, d\alpha$

$$= \int \frac{3/2 \cos\alpha\, d\alpha}{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 \sin^2\alpha}$$

$$= \int \frac{\cancel{3/2} \cos\alpha\, d\alpha}{\left(\frac{3}{2}\right)^{\cancel{2}} \cos^2\alpha}$$

$$= \frac{2}{3} \int \sec\alpha\, d\alpha.$$

$$= \frac{2}{3} \ln\left(\left|\sec\alpha + \tan\alpha\right|\right)$$

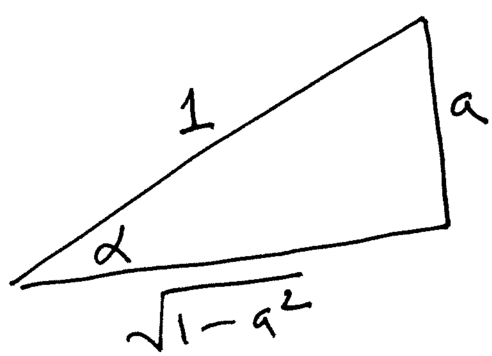
$$\therefore \int \frac{\sin\theta \, d\theta}{\cos^2\theta + \cos\theta - 2}$$

$$= \frac{2}{3} \ln \left| \sec\alpha + \tan\alpha \right|$$

where $\sin\alpha = \frac{2}{3} v$

$$= \frac{2}{3}\left(u + \frac{1}{2}\right) = \frac{2}{3}u + \frac{1}{3}$$

$$= \frac{2}{3}\cos\theta + \frac{1}{3} = a$$



$$\tan\alpha = \frac{a}{\sqrt{1-a^2}}$$

$$\cos\alpha = \sqrt{1-a^2}$$

$$\sec\alpha = \frac{1}{\sqrt{1-a^2}}$$

$$\sec\alpha + \tan\alpha = \frac{1+a}{\sqrt{1-a^2}}$$

$$= \left[\frac{4}{3} + \frac{2}{3}\cos\theta\right] \Big/ \sqrt{1 - \left[\frac{1}{3}(1+2\cos\theta)\right]^2}$$

$$= \frac{\frac{2}{3}\left[2+\cos\alpha\right]}{\sqrt{1-\left[\frac{1}{3}\left(1+2\cos\alpha\right)\right]^2}}$$

$$\therefore \int \frac{\sin\alpha \, d\alpha}{\cos^2\alpha+\cos\alpha-2}$$

$$= \frac{2}{3}\ln\left[\frac{\frac{2}{3}\left[2+\cos\alpha\right]}{\sqrt{1-\left[\frac{1}{3}\left(1+2\cos\alpha\right)\right]^2}}\right]$$

8(b) Ans

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$$\frac{x^4}{x^4+2x^2+1} = \frac{(x^4+2x^2+1)-(2x^2+1)}{x^4+2x^2+1}$$

$$= 1 - \frac{2x^2+1}{(x^2+1)^2}$$

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

$$= \frac{(Ax+B)(x^2+1)+(Cx+D)}{(x^2+1)^2}$$

$$= \frac{Ax^3+Ax+Bx^2+B+Cx+D}{(x^2+1)^2}$$

$$= \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(x^2+1)^2}.$$

Comparing co-efficients we have

$$\left.\begin{aligned} A &= 0 \\ B &= 2. \\ A+C &= 0 \\ B+D &= 1. \end{aligned}\right\} \Rightarrow \begin{aligned} &A=0,\ B=2 \\ &C=0. \\ &D=-1. \end{aligned}$$

$$\therefore \frac{2x^2+1}{(x^2+1)^2} = \frac{2}{x^2+1} + \frac{-1}{(x^2+1)^2}.$$

$$\therefore \frac{x^4}{(x^2+1)^2} = 1 - \frac{2}{x^2+1} + \frac{1}{(x^2+1)^2}$$

$$\int \frac{x^4\,dx}{(x^2+1)^2} = \int dx - 2\int \frac{dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2}.$$

$$= x - 2\tan^{-1}x + \int \frac{dx}{(x^2+1)^2}$$

similar to problem 6.

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$$\int \frac{dx}{(x^2+1)^2}$$

$$x = \tan\alpha$$

$$dx = \sec^2\alpha \, d\alpha.$$

$$= \int \frac{\sec^2\alpha \, d\alpha}{(1+\tan^2\alpha)^2}.$$

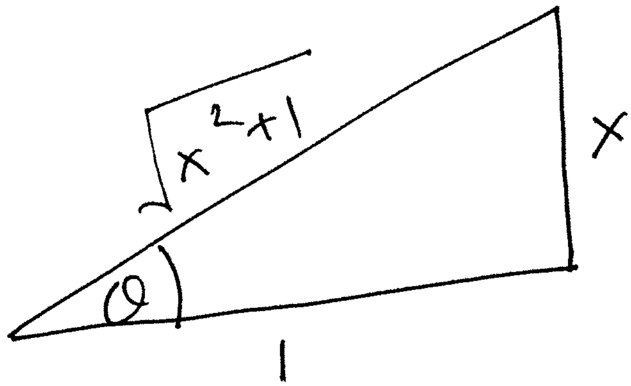
$$= \int \frac{\sec^2\alpha \, d\alpha}{\sec^4\alpha} = \int \cos^2\alpha \, d\alpha.$$

$$= \int \frac{1+\cos 2\alpha}{2} \, d\alpha$$

$$= \frac{1}{2}\alpha + \frac{1}{2}\frac{\sin 2\alpha}{2}.$$

$$= \frac{\alpha}{2} + \frac{1}{4}\sin 2\alpha.$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{4}\sin 2\alpha$$



$$\sin\theta = \frac{x}{\sqrt{x^2+1}} ; \quad \cos\theta = \frac{1}{\sqrt{x^2+1}}.$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{2x}{\sqrt{x^2+1}}.$$

$$\therefore \int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{\sqrt{x^2+1}}.$$

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$$\therefore \int \frac{x^4\,dx}{(x^2+1)^2}$$

$$= x - 2\tan^{-1}x$$

$$+ \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{\sqrt{x^2+1}}.$$

$$= x + \frac{1}{2}\frac{x}{\sqrt{x^2+1}} - \frac{3}{2}\tan^{-1}x.$$