Home Work II (solutions)

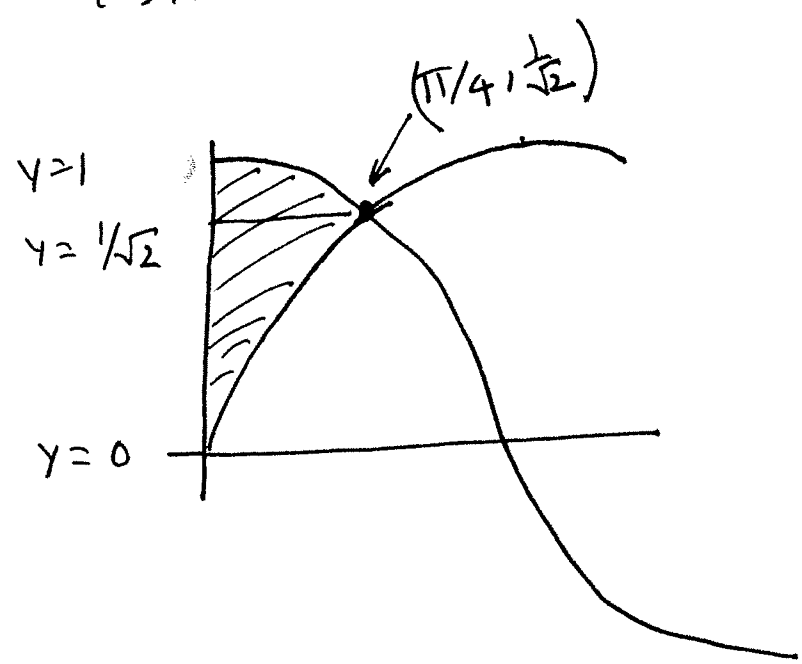
① Let us find the intersection between the two curves

$$y = \cos x \quad \text{and} \quad y = \sin x$$

$$\cos x = \sin x \implies x = \pi/4 \ \& \ \sin x = \cos x = 1/\sqrt{2}$$

To see this note that

$$\cos x = \sin x \implies \tan x = 1 \implies x = \tan^{-1} 1 = \pi/4 .$$



$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\,dx = \sin x + \cos x \Big|_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)$$

$$= \left(1/\sqrt{2} + 1/\sqrt{2}\right) - (0+1) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

$$\boxed{\text{Area} = \sqrt{2} - 1}$$

2 Area bounded by the curve $y=x^2$ and the line $y=4$ is given by

$$\int_{-2}^{2}(4-x^2)\,dx = 4x-\frac{x^3}{3}\Bigg|_{-2}^{2}$$

$$=\left(4\cdot 2-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)$$

$$=8-\frac{8}{3}+8-8/3$$

$$=16-\frac{16}{3}=\frac{48-16}{3}=\frac{32}{3}$$

Area bounded by the curve $y=x^2$ and the line $y=c$ is given by

$$\int_{-\sqrt{c}}^{\sqrt{c}}(c-x^2)\,dx = \left|\left(cx-\frac{x^3}{3}\right)\right|_{-\sqrt{c}}^{+\sqrt{c}}$$

$$=\left[c\sqrt{c}-\frac{(\sqrt{c})^3}{3}\right]-\left[-c\sqrt{c}+\frac{(\sqrt{c})^3}{3}\right]$$

$$=2c\sqrt{c}-\frac{2}{3}c\sqrt{c} = 4/3\, c\sqrt{c}$$

We want to calculate c such that

$$\frac{4}{3} c\sqrt{c} = \frac{1}{2}\,\frac{32}{3} = \frac{16}{3}$$

$$\Rightarrow c\sqrt{c} = 4$$

$$\Rightarrow c^{3/2} = 4$$

$$\Rightarrow c = (4)^{2/3} = (16)^{1/3} = \sqrt[3]{16}.$$

$$\boxed{c = \sqrt[3]{16}}.$$

(3)

Area =

$$\int_0^1 (1-\sqrt{x})^2\,dx$$

$$= \int_0^1 \left[1 + x - 2\sqrt{x}\right] dx$$

$$= x + \frac{x^2}{2} - 2x^{3/2}\cdot\frac{2}{3}\Big]_0^1$$

$$= x + \frac{x^2}{2} - \frac{4}{3}x^{3/2}\Big|_0^1$$

$$= 1 + \frac{1}{2} - \frac{4}{3} = \frac{6+3-8}{6}$$

$$= 1/6.$$

4 We imagine the sphere cut into thin slices by planes perpendicular to the x-axis. The volume of a typical slice between two planes at x and $x+\Delta x$ is approximately

$$\pi y^2 \Delta x = \pi(a^2 - x^2)\Delta x$$

and sum of all slices is approximately

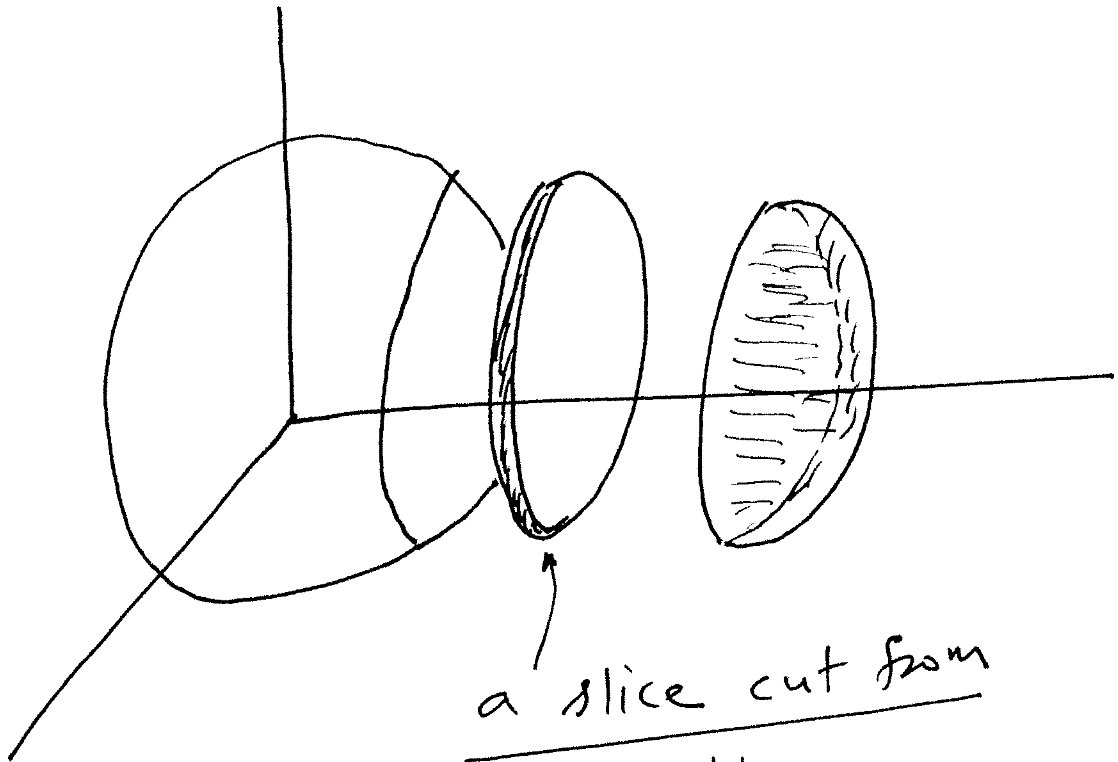
$$V_{-a}^{a} = \sum_{-a}^{a} \pi(a^2 - x^2)\Delta x.$$

The exact volume is given by

$$V_{-a}^{a} = \underset{\Delta x \to 0}{Lt} \sum_{-a}^{a} \pi(a^2 - x^2)\Delta x$$

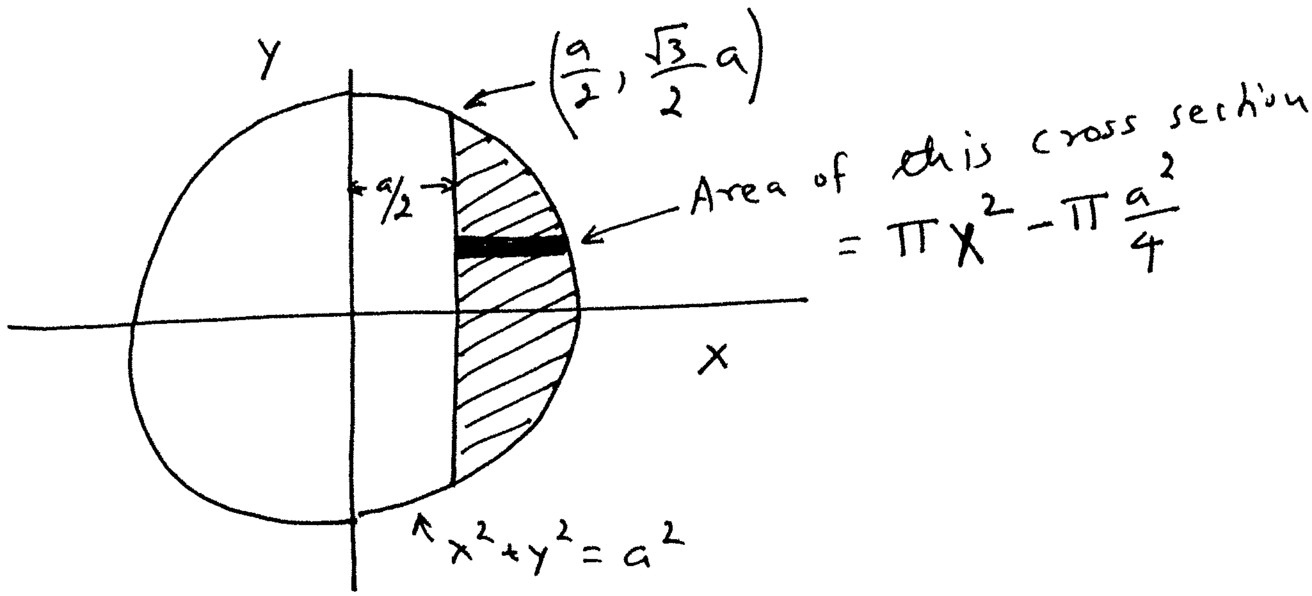
$$= \int_{-a}^{a} \pi(a^2 - x^2)\,dx$$

$$= \pi\left(a^2 x - \frac{x^3}{3}\right)\Bigg|_{-a}^{a} = \frac{4}{3}\pi a^3.$$



a slice cut from the sphere.

5 The volume in question could be generated by rotating about the y axis the area inside the circle $x^2+y^2=a^2$ lying to the right of the line $x=a/2$.



Volume of the cross section

$$\left(\pi x^2 - \pi \frac{a^2}{4}\right) \Delta y$$

$$= \left[\pi (a^2 - y^2) - \frac{\pi}{4} a^2\right] \Delta y$$

$$= \pi \left(\frac{3}{4} a^2 - y^2\right) \Delta y$$

Required volume

$$= \int_{-\frac{\sqrt{3}}{2}a}^{\sqrt{3}/2\, a} \left(\pi \frac{3}{4} a^2 - \pi y^2\right) dy$$

$$l = \frac{\sqrt{3}}{2} a$$

$$= \pi \left[\frac{3}{4} a^2 y - \frac{y^3}{3}\right]_{-l}^{l}$$

$$= \pi \left\{\left[\frac{3}{4} a^2 l - \frac{l^3}{3}\right] - \left[-\frac{3}{4} a^2 l + \frac{l^3}{3}\right]\right\}$$

$$= \pi \left[\frac{\cancel{6}^{\,3}}{\cancel{4}_{\,2}} a^2 l - \frac{2}{3} l^3\right]$$

$$= \pi \left[\frac{3}{2} a^2 \frac{\sqrt{3}}{2} a - \frac{2}{\cancel{3}} \frac{\cancel{3}\sqrt{3}}{8} a^3\right]$$

$$= \pi \left[\frac{3\sqrt{3}}{4} - \frac{1}{4}\sqrt{3}\right] a^3$$

$$= \pi \frac{\sqrt{3}}{2} a^3$$