Elementary Eigenvectors

My interest in this note is to talk about eigenvectors of a matrix:

Definition: Let A be a square $n \times n$ matrix and let λ be an eigenvalue of A, a non zero vector v in $\mathbb{R}^n$ is called an eigenvector of A is we have

$$Av = \lambda v.$$

Note that eigenvectors are not unique. If v is an eigenvector, then $2v$ is also an eigenvector.

Good News: For any matrix A and any eigenvalue λ there always exist at least one eigenvector.

Example 0: Let us choose A to be the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$$

The characteristic polynomial of A is obtained by computing

$$det(\lambda I - A) = \lambda^3 - c\lambda^2 - b\lambda - a$$

The roots of the above polynomial are at λ_1, λ_2 and λ_3 which are the three eigenvalues of the matrix A. Let

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

be an eigenvector of A corresponding to the eigenvalue λ_1. We need to solve the equation

$$A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

It follows that

$$v_2 = \lambda_1 v_1,$$

$$v_3 = \lambda_1 v_2 = \lambda_1^2 v_1$$

and

$$av_1 + bv_2 + cv_3 = \lambda_1 v_3.$$

We obtain

$$av_1 + b\lambda_1 v_1 + c\lambda_1^2 v_1 = \lambda_1^3 v_1.$$

The last equation is always satisfied, since

$$\lambda_1^3 = a + b\lambda_1 + c\lambda_1^2.$$

(recall that λ_1 is an eigenvalue and hence satisfies the characteristic polynomial). The eigenvector is given by

$$\begin{pmatrix} 1 \\ \lambda_1 \\ \lambda^2 \end{pmatrix} v_1$$

where v_1 can be thought as a scale factor. Upto a scale factor an eigenvector of the matrix A for the eigenvalue λ_1 is given by

$$\begin{pmatrix} 1 \\ \lambda_1 \\ \lambda^2 \end{pmatrix}.$$

When A is the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}$$

the three eigenvalues of A are at -1, -2 and -3. The corresponding eigenvectors are at

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix},$$

respectively. You can also use 'matlab' to calculate eigenvalues. The command is

[v1,v2] = eig(A).

The matlab session has been attached at the end.

Example 1: Let us choose A to be the matrix

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 7 \\ -1 & 4 & -3 \end{pmatrix}$$

The characteristic polynomial of A is obtained by computing

$$det(\lambda I - A) = \lambda^3 - 5\lambda^2 - 47\lambda - 6$$

The roots of the above polynomial are at 9.8389, -4.7094 and -0.1295 which are the three eigenvalues of the matrix A. Note that all three eigenvalues are real.

The eigenvectors for this matrix is harder to compute because the eigenvalues are real. So we resort to matlab (see attached).

Example 2: Let us choose A to be the matrix

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 8 & 3 & 6 \\ 0 & -6 & 3 \end{pmatrix}$$

The characteristic polynomial of A is obtained by computing

$$det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 75\lambda - 177$$

The roots of the above polynomial are at $3.6581 + 5.8879i$, $3.6581 - 5.8879i$ and 3.6838 which are the three eigenvalues of the matrix A. Note that all the eigenvalues are not real, but the eigenvalues occur in complex conjugate pairs.

Once again the eigenvectors are hard to compute by hand. So we resort to matlab (see attached).
Let us now look at a sufficiently complicated example:

Example 3: Let us choose A to be the matrix

$$A = \begin{pmatrix} 5 & 0 & 1 & 1 & 0 & 0 \\ 8 & 8 & 6 & 0 & 1 & 0 \\ 0 & -6 & 8 & 0 & 0 & 1 \\ 1 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 8 & 3 & 6 \\ 0 & -1 & 0 & 3 & -6 & 3 \end{pmatrix}$$

The characteristic polynomial of A is obtained by computing

$$det(\lambda I - A) = \lambda^6 - 32\lambda^5 + 482\lambda^4 - 4078\lambda^3 + 21054\lambda^2 - 60643\lambda + 7$$

The roots of the above polynomial are at $8.4927 + 6.3631i$, $8.4927 - 6.3631i$, $3.6075 + 5.5606i$, $3.6075 - 5.5606i$, 4.9295 and 2.8701 which are the six eigenvalues of the matrix A. Note that all the eigenvalues are not real, but the eigenvalues occur in complex conjugate pairs.

```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> A=[0 1 0;0 0 1;-6 -11 -6]

A =

     0     1     0
     0     0     1
    -6    -11    -6

>> [v1,v2]=eig(A)

v1 =

   -0.5774    0.2182   -0.1048
    0.5774   -0.4364    0.3145
   -0.5774    0.8729   -0.9435

v2 =

   -1.0000         0         0
         0   -2.0000         0
         0         0   -3.0000

>>
```

The First Eigenvalue and the
corresponding Eigenvector

The Eigenvalues and the Eigenvectors of Example 0

```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> A=[2 4 5;3 6 7;-1 4 -3]

A =
```

Example One

```
     2      4      5
     3      6      7
    -1      4     -3
```

EIGENVALUES and
EIGENVECTORS

```
>> [v1,v2]=eig(A)

v1 =

   -0.3523    -0.9287     0.5473
   -0.4409     0.0328     0.8102
    0.8255     0.3693     0.2098


v2 =

   -4.7094          0          0
        0    -0.1295          0
        0          0     9.8389
```

```
>> A=[5 0 1;8 3 6;0 -6 3]

A =
```

Example Two

```
     5      0      1
     8      3      6
     0     -6      3
```

The eigenvalues and eigenvectors are

```
>> [v1,v2]=eig(A)

v1 =
```

complex conjugates

```
  -0.6025                 -0.0260 - 0.1141i   -0.0260 + 0.1141i
  -0.0904                 -0.0775 - 0.6935i   -0.0775 + 0.6935i
   0.7930                  0.7067              0.7067


v2 =

   3.6838                        0                      0
        0             3.6581 + 5.8879i               0
        0                        0          3.6581 - 5.8879i
```

```
>> I=[1 0 0;0 1 0;0 0 1];
>> Z=[0 0 0;0 0 0;0 0 0];
>> A=[A I;Z A]

A =

     5      0      1      1      0      0
     8      3      6      0      1      0
```

```
     0    -6     3     0     0     1
     0     0     0     5     0     1
     0     0     0     8     3     6
     0     0     0     0    -6     3

>> A(4,1)=1;
>> A(6,2)=-1
```

Example Three

```
A =

     5     0     1     1     0     0
     8     3     6     0     1     0
     0    -6     3     0     0     1
     1     0     0     5     0     1
     0     0     0     8     3     6
     0    -1     0     0    -6     3

>> [v1,v2]=eig(A)

v1 =

  Columns 1 through 4

  -0.4240              0.4164         0.0197 + 0.0939i   0.0197 - 0.0939i
  -0.0607              0.0688         0.0582 + 0.5867i   0.0582 - 0.5867i
   0.5724             -0.5596        -0.5954            -0.5954
   0.4199              0.4386         0.0510 - 0.0601i   0.0510 + 0.0601i
  -0.0216              0.1435         0.2442 - 0.3039i   0.2442 + 0.3039i
  -0.5587             -0.5439         0.2159 + 0.2897i   0.2159 - 0.2897i

  Columns 5 through 6

  -0.0207 - 0.0942i  -0.0207 + 0.0942i
  -0.0594 - 0.5852i  -0.0594 + 0.5852i
   0.5956             0.5956
   0.0212 - 0.0460i   0.0212 + 0.0460i
   0.2429 - 0.2625i   0.2429 + 0.2625i
   0.2937 + 0.2707i   0.2937 - 0.2707i
```

Two Real Eigenvectors

Two pairs of complex conjugate eigenvectors

```
v2 =

  Columns 1 through 4

   2.6598              0                  0                  0
        0         4.7094                  0                  0
        0              0         3.2239 + 5.4264i           0
        0              0                  0         3.2239 - 5.4264i
        0              0                  0                  0
        0              0                  0                  0

  Columns 5 through 6

        0              0
        0              0
        0              0
        0              0
        0              0
   4.0915 + 6.3494i    0
```
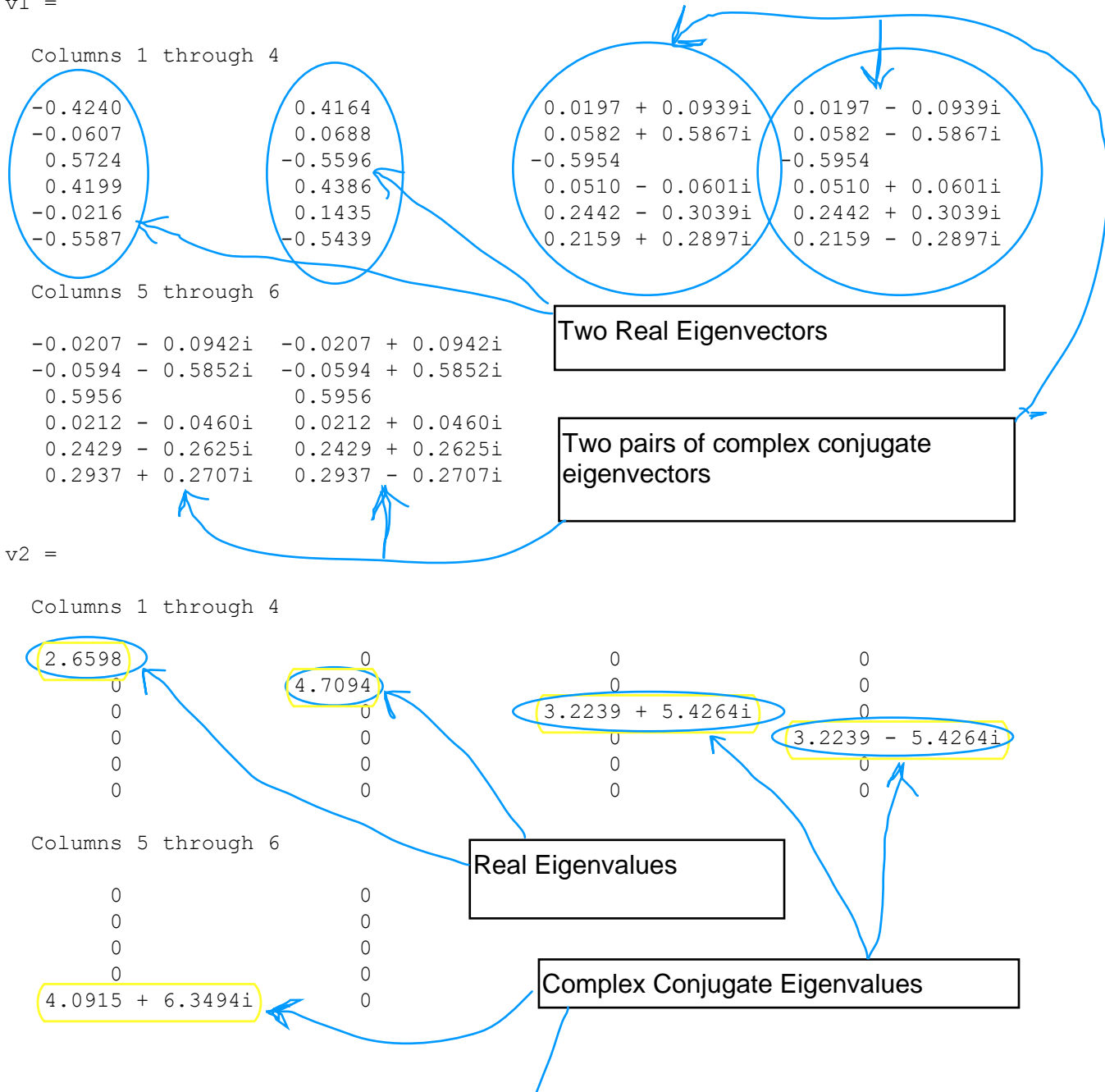
Real Eigenvalues

Complex Conjugate Eigenvalues

```
       0              4.0915 - 6.3494i

>>
```

Typically we are not happy finding just one eigenvector.

We would rather like to find a set of n linearly independent eigenvectors, possibly corresponding to different eigenvalues.

If a $n \times n$ matrix A has n distinct eigenvalues then each of the eigenvalues have at least one eigenvector and all the eigenvalues together have n linearly independent eigenvectors.

Theorem: Assume that v_i is an eigenvector of A corresponding to eigenvalue λ_i then v_i and v_j, for $i \neq j$ are linearly independent if $\lambda_i \neq \lambda_j$.

Proof: From the definition of eigenvectors it follows that

$$Av_i = \lambda_i v_i.$$

Assume that v_i and v_j are not independent, it follows that there exists δ such that

$$v_i = \delta v_j,$$

which implies that

$$Av_i = \delta A v_j.$$

Thus we conclude that

$$\lambda_i v_i = \delta \lambda_j v_j,$$

which implies that

$$v_i = \delta \frac{\lambda_j}{\lambda_i} v_j = \delta v_j.$$

Cancelling δ we have $\frac{\lambda_j}{\lambda_i} = 1$, which is a contradiction because the eigenvalues are distinct.

Thus our original assumption, "Assume that v_i and v_j are not independent" must be wrong.

If v_1, $v_2 \cdots v_n$ is a set of n independent eigenvectors of a matrix A, we can define a matrix V as follows:

$$V = [v_1, v_2, \cdots, v_n]$$

where V is invertible because the eigenvectors are independent. We can show that

$$V^{-1}AV = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

The operation on the left is called "Similarity Transformation". The matrix on the right is a "Diagonal Matrix".

For reasons that we do not know yet, diagonal matrices are very important class of matrices. What we have seen thus far is that if a matrix has distinct eigenvalues then there exist a similarity transformation which diagonalizes the matrix.

At this point, some of the questions that linger in our mind are as follows:
   Question 1: Why diagonalize matrices at all (by similarity transformation)??
   Question 2: When can we diagonalize matrices by similarity transformation??
   Question 3: If we cannot diagonalize a matrix by similarity transformation, what is the next best thing we can do??

If you go back and look at the matlab sessions, you will notice that all the matrices have distinct eigenvalues. The V matrix of eigenvectors are provided by the matrix v1. The only difficulty at this point is that the eigenvalues and therefore the eigenvectors can be complex. Hence the diagonal matrix can be complex. This is the case in example 2 and 3.