ESE501 Solutions to the Midterm II

1. In this problem we are considering a first order ordinary differential equation given by

$$\dot{y}(t) \;+\; 3\,y(t) \;=\; 5\,u(t),$$

where

$$y(0) \;=\; 2,$$

and where

$$u(t) \;=\; e^{-7t},\; t \;\geq\; 0.$$

Calculate y(t) manually by showing all the steps.

Solution: Using the variation of constants formula we obtain

$$y(t) \;=\; e^{-3t}\,y(0) \;+\; \int_0^t e^{-3(t-\tau)}\,5\,e^{-7\tau}\,d\tau \;=$$

$$2\,e^{-3t} \;+\; 5\,e^{-3t}\int_0^t e^{-4\tau}\,d\tau \;=$$

$$2\,e^{-3t} \;-\; \frac{5}{4}\,e^{-3t}(e^{-4t} \;-\; 1) \;=$$

$$(2 \;+\; \frac{5}{4})\,e^{-3t} \;-\; \frac{5}{4}\,e^{-7t}.$$

It follows that

$$y(t) \;=\; \frac{13}{4}\,e^{3t} \;-\; \frac{5}{4}\,e^{-7t}.$$

2. A 2×2 matrix A has repeated eigenvalues at 2, 2, with a corresponding chain of generalized eigenvectors v_1, v_2 where

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } v_2 = \begin{pmatrix} 7 \\ 4 \end{pmatrix}.$$

Assume that v_1 is the eigenvector and v_2 is the generalized eigenvector.

Calculate e^{At} from this data.

Solution: Using the eigenvectors and generalized eigenvectors of the matrix A, define the matrix P as follows:

$$P = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}.$$

It is well known that $P^{-1}\ A\ P$ has the jordan canonical form

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},$$

where

$$e^{Bt} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

It would follow that

$$e^{At} = P\ e^{Bt}\ P^{-1},$$

i.e.

$$e^{At} = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}.$$

which equals

$$e^{At} = \begin{pmatrix} e^{2t}(1-2t) & 4te^{2t} \\ -te^{2t} & e^{2t}(1+2t) \end{pmatrix}.$$

3. Let us define the following 2×2 matrices:

$$B = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}, \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate

$$\left(\frac{2B + I}{5}\right)^{100}$$

Solution: Writing

$$(2B + I)^{100} = \alpha_0 I + \alpha_1 B,$$

we obtain

$$(2\lambda + 1)^{100} = \alpha_0 + \alpha_1 \lambda,$$

where λ the eigenvalue is at 2. Since the eigenvalues are repeating, we take a single derivative w.r.t. λ and obtain

$$200(2\lambda + 1)^{99} = \alpha_1.$$

Solving for the coefficients α_0 and α_1 we obtain

$$\alpha_1 = 200(5)^{99}, \ \alpha_0 = -395(5)^{99}.$$

It follows that

$$(2B + I)^{100} = \begin{pmatrix} -395 & 200 \\ -800 & 405 \end{pmatrix} (5)^{99} =$$

$$\begin{pmatrix} -79 & 40 \\ -160 & 81 \end{pmatrix} (5)^{100}.$$

Thus we conclude that

$$\left(\frac{2B + I}{5}\right)^{100} = \begin{pmatrix} -79 & 40 \\ -160 & 81 \end{pmatrix}$$

4. A discrete time recursive system is given by

$$X_{k+1} = A\,X_k + b\,u_k,\ y_k = c\,X_k,$$

where $X_0 = 0$. The matrices are given by

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$c = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are at $\frac{1}{2}$ and $\frac{1}{4}$. The input sequence u_k is given by

$$u_k = \{1, 1, 1, \ldots\}$$

Calculate the sequence y_k given by

$$y_k = \sum_{j=1}^{k} c\,A^{j-1}\,b.$$

Solution: Let us denote

$$S = I + A + \ldots + A^{k-1}$$

and we compute

$$S = (I - A^k)(I - A)^{-1}.$$

It follows that

$$y_k = c(I - A^k)(I - A)^{-1}b$$

where

$$(I - A)^{-1}b = \begin{pmatrix} \frac{8}{3} \\ \\ \frac{8}{3} \end{pmatrix}.$$

Let us now write

$$A^k = \alpha_0 I + \alpha_1 A,$$

it follows that

$$c(I - A^k) = (1 - \alpha_0 \quad - \alpha_1).$$

We would thus obtain

$$y_k = \frac{8}{3}\,(1 - (\alpha_0 + \alpha_1)).$$

The quantity $\alpha_0 + \alpha_1$ is compute as follows:

If λ_1 and λ_2 are the two eigenvalues of A we write:

$$\begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \end{pmatrix}.$$

Solving, we obtain

$$\alpha_0 = \frac{\lambda_1^k \lambda_2 - \lambda_2^k \lambda_1}{\lambda_2 - \lambda_1}$$

and

$$\alpha_1 = \frac{\lambda_2^k - \lambda_1^k}{\lambda_2 - \lambda_1}.$$

It would follow that

$$\alpha_0 + \alpha_1 = \frac{(\lambda_2 - 1)\lambda_1^k - (\lambda_1 - 1)\lambda_2^k}{\lambda_2 - \lambda_1}$$