ESE501 Midterm II

There are four questions of equal weightage. The exam is open book and notes. Calculators and computers are not allowed. Total time allotted is 90 minutes.

1. In this problem we are considering a first order ordinary differential equation given by

$$\dot{y}(t) \; + \; 3\, y(t) \; = \; 5\, u(t),$$

where

$$y(0) \; = \; 2,$$

and where

$$u(t) \; = \; e^{-7t}, \; t \; \geq \; 0.$$

Calculate y(t) manually by showing all the steps.

2. A 2×2 matrix A has repeated eigenvalues at 2, 2, with a corresponding chain of generalized eigenvectors v_1, v_2 where

$$v_1 \; = \; \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } v_2 \; = \; \begin{pmatrix} 7 \\ 4 \end{pmatrix}.$$

Assume that v_1 is the eigenvector and v_2 is the generalized eigenvector.

Calculate e^{At} from this data.

Algebraic fact:

$$\begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}^{-1} \; = \; \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}.$$

3. Let us define the following 2×2 matrices:

$$B \; = \; \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}, \text{ and } I \; = \; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate

$$\left(\frac{2B + I}{5} \right)^{100}$$

Fact: The eigenvalues of the matrix B are at 2, 2.

4. A discrete time recursive system is given by

$$X_{k+1} = A X_k + b u_k, \; y_k = c X_k,$$

where $X_0 = 0$. The matrices are given by

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$c = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are at $\frac{1}{2}$ and $\frac{1}{4}$. The input sequence u_k is given by

$$u_k = \{1, 1, 1, \ldots\}$$

Calculate the sequence y_k given by

$$y_k = \sum_{j=1}^{k} c \, A^{j-1} \, b.$$

You can leave the answer as a power of the eigenvalues

Algebraic fact:

$$(I - A)^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{8}{3} \\ -\frac{1}{3} & \frac{8}{3} \end{pmatrix}$$

THE END