ESE501: First Midterm Exam

There are 20 multiple choice questions. Please circle one appropriate answer. Total time of the exam is 90 minutes after which you must stop working. The exam is open notes and books but calculators and computers are strictly not allowed. There is no discredit for wrong answer, so you can guess.

GOOD LUCK

Your Name:

ID Number:

Your Score

1) The **cross product** of two vectors v_1 and v_2 in $\mathbb{R}^3$ is

a) a vector v_3 which crosses both the vectors v_1 and v_2.

b) a vector v_3 such that v_1, v_2 and v_3 form a triangle.

c) a vector v_3 which is parallel to v_1 and v_2.

d) a vector v_3 which is of unit length and is perpendicular to both v_1 and v_2.

e) None of the above

2) The best description of a **homogeneous line** in $\mathbb{R}^2$ is the following

a) Ink spreads homogeneously when you draw it.

b) Thickness of the line is always the same.

c) The line always passes through the origin.

d) The line never touches the origin.

e) A line of similar objects.

3) **Angle** between the two vectors

$$v_1 = (2 \;\; 3 \;\; 2 \;\; 1)$$

and

$$v_2 = (1 \;\; 0 \;\; 1 \;\; -1)$$

in $\mathbb{R}^4$

a) can be computed, but who will do it?

b) does not make sense.

c) can be computed and is given by $\cos^{-1}(3/54)$.

d) can be computed and is given by $\cos^{-1}(3/\sqrt{54})$.

e) can be computed and is given by $\cos^{-1}\sqrt{(3/54)}$.

4) **Area** of a triangle in $\mathbb{R}^4$ with corners located at v_1, v_2 and the origin (where v_1 and v_2 are given in problem 3)

a) does not make sense because we are in $\mathbb{R}^4$.

b) perhaps make sense but make me do it.

c) does make sense and is given by

$$\frac{1}{2}\ \|v_1 \times v_2\|. \ \ (\times \text{ stands for cross product})$$

d) does make sense and is given by

$$\frac{1}{2}\ \|v_1\|\ \|v_2 - \text{proj}_{[v_1]} v_2\|.$$

e) none of the above.

5) The **line**

$$\frac{x - x_0}{h_1} = \frac{y - y_0}{h_2} = \frac{z - z_0}{h_3}$$

in $\mathbb{R}^3$ with coordinates (x, y, z)

a) is always perpendicular to the vector $(h_1 \;\; h_2 \;\; h_3)$.

b) is always parallel to the vector $(h_1 \;\; h_2 \;\; h_3)$.

c) is always perpendicular to the vector $(x_0 \;\; y_0 \;\; z_0)$.

d) is always parallel to the vector $(x_0 \;\; y_0 \;\; z_0)$.

e) is always contained in a plane spanned by the two vectors $(h_1 \;\; h_2 \;\; h_3)$ and $(x_0 \;\; y_0 \;\; z_0)$.

6) In $\mathbb{R}^4$,

a) any four vectors would form a **basis** of $\mathbb{R}^4$.

b) any five vectors would not form a basis of $\mathbb{R}^4$.

c) any three linearly independent vectors would form a basis of $\mathbb{R}^4$.

d) any four linearly dependent vectors would form a basis of $\mathbb{R}^4$.

e) none of the above.

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7) The **coordinates** of the vector $(3\ \ 5)$ with respect to the basis

$$B = \{(1\ \ 3), (-1\ \ 2)\}$$

in $\mathbb{R}^2$

a) is given by 3, 5.

b) cannot be computed because it is not unique.

c) is given by 1, 3.

d) given by –1, 2.

e) none of the above.

8) **Span** of the two linearly independent vectors

$$v_1 = (\alpha_1\ \ \beta_1\ \ \gamma_1) \text{ and } v_2 = (\alpha_2\ \ \beta_2\ \ \gamma_2)$$

in $\mathbb{R}^3$

a) describes a region in $\mathbb{R}^3$ between two intermediate supports of a bridge.
b) is a pair of points in $\mathbb{R}^3$ usually denoting the tip of the thumb and the tip of the little finger when the hand is fully extended.
c) the distance between the tips of the wings of an airplane.
d) is a plane in $\mathbb{R}^3$ given by the equation

$$(\beta_1\gamma_2 + \beta_2\gamma_1)\mathrm{x} + (\alpha_2\gamma_1 + \alpha_1\gamma_2)\mathrm{y}$$
$$+(\alpha_1\beta_2 + \alpha_2\beta_1)\mathrm{z} = 0$$

with coordinates x,y and z.

e) none of the above.

9) The set of points $(x\ \ \ y\ \ \ z)$ in $\mathbb{R}^3$ whose coordinates add up to zero

a) is a **vector space** of dimension 3.

b) is not a vector space.

c) is a vector space with basis given by $\{(1\ \ 0\ \ 1), (0\ \ 1\ \ 1)\}$.
d) is a vector space spanned by

$$\{(1\ \ 0\ \ -1), (0\ \ 1\ \ -1)\}.$$

e) none of the above.

10) In $\mathbb{R}^8$ with coordinates $x_1, x_2, \ldots, x_8$ the vector

$$v = (1, 0, 0, 0, 0, 0, 0, 1)$$

a) cannot be **projected** on the plane $\mathbf{P}$ described by

$$\sum_{i=1}^{8} x_i = 0.$$

b) can be projected on the plane $\mathbf{P}$ but I am not going to do it.
c) can be projected on the plane $\mathbf{P}$ and the projection is

$$\begin{pmatrix} \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{pmatrix}.$$

d) can be projected on the plane $\mathbf{P}$ and the projection is

$$\begin{pmatrix} \frac{-3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{-3}{4} \end{pmatrix}.$$

e) can be projected on the plane $\mathbf{P}$ and the projection is

$$\begin{pmatrix} \frac{-3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \end{pmatrix}.$$

11) Let us consider the following three vectors $v_1,\ v_2$ and v_3 in $\mathbb{R}^3$.

$$v_1 = (1\ \ \ 0\ \ \ -2),$$

$$v_2 = (-1\ \ \ 0\ \ \ 2),$$

$$v_3 = (1\ \ \ 0\ \ \ 2).$$

a) The three vectors $v_1,\ v_2$ and v_3 are **linearly independent**.

b) Every pair of vectors $\{v_1,\ v_2\}$, $\{v_2,\ v_3\}$, $\{v_1,\ v_3\}$, are linearly independent.

c) All pairs except the pair $\{v_1,\ v_2\}$ are linearly independent.

d) All pairs except the pair $\{v_1,\ v_3\}$ are linearly independent.

e) none of the above.

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12) The **determinant** of the matrix

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & 1 \\ 3 & 5 & 900 \end{pmatrix}$$

is given by

a) 900

b) 0

c) 4

d) –4

e) 5

13) Using **Cramer's rule** to solve the set of equations

$$x + 3y + 5z = 1, z = 3, 3x + 5y + 900z = 4,$$

we can write

$x = \Delta / det$ A, (matrix A given in problem 12)

where

a) Δ = –8018

b) Δ = 8032

c) Δ = –8032

d) Δ = 8018

e) None of the above.

14) The **characteristic polynomial** $p(\lambda)$ of a 2×2 matrix B is given by

$$p(\lambda) = \lambda^2 - 3\lambda.$$

Using **Cayley Hamilton Theorem**, we can then say that

a) $B^n = 3^{n-1}B.$

b) $B^n = 3^{n+1}B.$

c) $B^n = 3^n B.$

d) $B^n = B^{98}.$

e) none of the above.

15) The **Adjoint** of the matrix

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

is given by

a) $$\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$$

b) $$\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

c) $$\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

d) $$\begin{pmatrix} 2 & -1.5 \\ -1 & .5 \end{pmatrix}$$

e) $$\begin{pmatrix} 2 & -1 \\ -1.5 & .5 \end{pmatrix}.$$

16) The **rank** and **nullity** of the matrix

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{pmatrix}$$

a) are 2 and 3 respectively.

b) are 3 and 2 respectively.

c) are 1 and 2 respectively.

d) are 2 and 1 respectively.

e) cannot be determined from this data.

17) A **basis of the null space** of the matrix

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{pmatrix}$$

is given by

a) $\{(-2, 1, 0), (-5, 0, 1)\}.$

b) $\{(-2, 1, 0)\}.$

c) $\{(-5, 0, 1)\}.$

d) $\{(-2, 1, 0), (-5, 0, 1), (-7, 1, 1)\}.$

e) $\{(2, 1, 0), (5, 0, 1)\}.$

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18) The **eigenvalues** of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a) are all at 0.

b) are at 0, 1 and –1

c) are all at 1

d) are at 0, 1 and 1

e) cannot be determined for sure.

19) Let C be a 2×3 matrix and D be a 3×3 matrix.

a) The **product** CD can be defined but DC cannot be defined.
b) The product DC can be defined but CD cannot be defined.
c) Both the products CD and DC can be defined and they are equal.
d) Both the products CD and DC can be defined and they are not equal.
e) None of the products CD and DC can be defined.

20) The **product** FG of the two matrices F and G given by

$$F = \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix} \text{ and } G = \begin{pmatrix} 3 & 9 \\ 5 & 0 \end{pmatrix}$$

is the following

a)
$$\begin{pmatrix} 24 & 18 \\ 5 & 1 \end{pmatrix}.$$

b)
$$\begin{pmatrix} 27 & 18 \\ 5 & 0 \end{pmatrix}.$$

c)
$$\begin{pmatrix} 26 & 18 \\ 4 & 0 \end{pmatrix}$$

d)
$$\begin{pmatrix} 26 & 17 \\ 5 & 0 \end{pmatrix}$$

e)
$$\begin{pmatrix} 26 & 18 \\ 5 & 0 \end{pmatrix}$$

f)
$$\begin{pmatrix} 24 & 18 \\ 6 & 0 \end{pmatrix}$$

THANKS: WE ARE DONE

Question 1: **Correct answer is (e)**

a) It is absurd. Remember that vectors are points in a space.
b) Cross product is a vector perpendicular, so they cannot form a triangle.
c) Cross product is a vector perpendicular, so it cannot be parallel to the two vectors.
d) This answer is close but a cross product is not an unit vector.

Question 2: **Correct answer is (c)**

The choices a) b) and e) are absurd.
d) If a line does not touch the origin, it is called an affine line.

Question 3: **Correct answer is (d)**

Angle between any two vectors in $\mathbb{R}^n$ does make sense. It is computed by the formula

$$v_1.v_2 = \|v_1\|\|v_2\|cos\theta.$$

i.e.

$$\theta = cos^{-1}\frac{v_1.v_2}{\|v_1\|\|v_2\|}.$$

Question 4: **Correct answer is (d)**

Area of a triangle is $\frac{1}{2}.\text{base}.\text{height}$. The choice c) is close and would have been the right answer in $\mathbb{R}^3$ but the cross product does not make sense in $\mathbb{R}^4$.

Question 5: **Correct answer is (b)**

The line passes through the point (x_0, y_0, z_0) and is parallel to the vector (h_1, h_2, h_3).

Question 6: **Correct answer is (b)**

a) Any four linearly independent vectors would form a basis of $\mathbb{R}^4$.
b) Any five vectors in $\mathbb{R}^4$ would always be linearly dependent and cannot form a basis of $\mathbb{R}^4$.
c) Three linearly independent vectors would not be able to span the entire $\mathbb{R}^4$, hence cannot form a basis.
d) The vectors have to be independent.

Question 7: **Correct answer is (e)**

The choice b) is incorrect because every vector would have a coordinate with respect to any basis.
The choices a) c) d) are incorrect because they don't match with the answer.

If α and β are the coordinates, we have

$$\alpha - \beta = 3, 3\alpha + 2\beta = 5;$$

which implies that $\beta = -\frac{4}{5}$ and $\alpha = \frac{11}{5}$.

Question 8: **Correct answer is (e)**

Span of a set of vectors $v_1, \cdots, v_n$ is the set of all vectors that are linear combinations of the vectors $v_1, \cdots, v_n$. The span of one non zero vector is a homogeneous line. The span of two linearly independent vectors is a homogeneous plane and so on.

Choices a), b), c) are therefore clearly wrong. The choice d) is close but does not quite describe the required plane (there is a sign error).

Question 9: **Correct answer is (d)**

In this problem we have

$$x + y + z = 0.$$

The vectors are of the form

$$(x, y, -x - y)$$

which would be spanned by $(1, 0, -1)$ and $(0, 1, -1)$. Thus this space is a plane with dimension 2. It is also a vector space since **'span of any set of vectors is always a vector space'.**

Choices a) and b) are therefore clearly wrong.
The choice c) is wrong because the coordinates of the vectors don't add up to zero.

Remark: Try this problem again if

$$x + y + z = 1.$$

In this case the space would be an affine plane which is not a vector space.

Question 10: **Correct answer is (c)**

a) Any vector can be projected on the plane P, since P is a homogeneous plane, which is a vector space.
b) is almost correct but does not give you any credit.
The choices d) and e) are the wrong projections.

Remember that if the projection of v is the vector w, then $v - w$ is perpendicular to all the vectors in P, in particular $v - w$ is perpendicular to all the basis vectors in P. One choice of a basis would be

$(1, -1, 0, 0, 0, 0, 0, 0)$,
$(1, 0, -1, 0, 0, 0, 0, 0)$,
$(1, 0, 0, -1, 0, 0, 0, 0)$,
$(1, 0, 0, 0, -1, 0, 0, 0)$,
$(1, 0, 0, 0, 0, -1, 0, 0)$,
$(1, 0, 0, 0, 0, 0, -1, 0)$,
$(1, 0, 0, 0, 0, 0, 0, -1)$.

Question 11: **Correct answer is (c)**

We have $v_1 = -v_2$. Thus v_1 and v_2 are linearly dependent. It can be checked that v_1 and v_3 are linearly independent. Take a linear combination and set it equal to zero. We get $\alpha + \beta = 0$, and $-2\alpha + 2\beta = 0$. The only solution is the trivial solution.

Question 12: **Correct answer is (c)**

Expand the determinant with respect to the second row. We obtain:

$$det\ A\ =\ -1\ det\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} = (-1)(5-9) = 4.$$

Question13: **Correct answer is (a)**

The equation can be written as

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & 1 \\ 3 & 5 & 900 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

Cramer's rule says that

$$x = \frac{det\ A_1}{det\ A}$$

where

$$A_1 = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 0 & 1 \\ 4 & 5 & 900 \end{pmatrix}.$$

Question 14: **Correct answer is (a)**

Cayley Hamilton Theorem says that

$$B^2\ =\ 3B.$$

It follows that

$$B^3\ =\ 3^2B,$$
$$B^4\ =\ 3^3B,$$

and in general

$$B^n\ =\ 3^{n-1}B.$$

Question 15: **Correct answer is (c)**

If

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

The cofactor matrix is given by $[c_{ij}]$, where $c_{ij} = (-1)^{i+j}\ \det A_{ij}$. A_{ij} is the sub-matrix of A obtained by deleting the i^{th} row and the j^{th} column.

We have $c_{11} = 4$, $c_{12} = -2$,, $c_{21} = -3$, , $c_{22} = 1$. The cofactor matrix is given by

$$C = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

The adjoint matrix is the transpose of the cofactor matrix which can now be readily computed.

Question 16: **Correct answer is (c)**

Since rank + nullity is the number of columns of the matrix, the choices a) and b) are clearly wrong.

Since the second row is twice the first row, there is only one linearly independent row in this matrix. Hence rank is 1. Therefore nullity must be 2.

Question 17: **Correct answer is (a)**

The null space is described by the equation

$$x + 2y + 5z = 0.$$

The above equation is obtained by row reducing the matrix to obtain

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

The vectors in the null space are of the form $(-2y-5z, y, z)$ which are linear combinations of linearly independent vectors $(-2, 1, 0)$ and $(-5, 0, 1)$.

Question 18: **Correct answer is (b)**

The characteristic polynomial of the matrix is given by

$$\lambda^3 + 0.\lambda^2 + (-1).\lambda^1 + 0 \;=\; \lambda^3 - \lambda.$$

Note that the matrix is in the standard form for which the characteristic polynomial can be written by inspection. The roots of the above polynomial can also be readily obtained.

Question 19: **Correct answer is (a)**

Recall that the product CD is defined if the number of columns of C equals the number of rows of D. Likewise the product DC is defined if the number of columns of D equals the number of rows of C. In our problem, CD is defined but DC is not.

Question 20: **Correct answer is (e)**

Can be checked by multiplying the matrices.