Solutions to the First Midterm Exam

1. $$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \alpha_3 \begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \alpha_1 + \alpha_2 \lambda_1 + \alpha_3 \lambda_1^2 = 0 \quad (A)$$

$$\alpha_1 + \alpha_2 \lambda_2 + \alpha_3 \lambda_2^2 = 0 \quad (B)$$

$$\alpha_1 + \alpha_2 \lambda_3 + \alpha_3 \lambda_3^2 = 0 \quad (C)$$

Subtracting (B) from (A) we obtain

$$\alpha_2 (\lambda_1 - \lambda_2) + \alpha_3 (\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2) = 0$$

$$\Rightarrow \alpha_2 + \alpha_3 (\lambda_1 + \lambda_2) = 0 \quad (D)$$

$$(\because \lambda_1 - \lambda_2 \neq 0)$$

Likewise, subtracting (C) from (A) we obtain

$$\alpha_2 + \alpha_3 (\lambda_1 + \lambda_3) = 0 \quad (E)$$

$$(\because \lambda_1 - \lambda_3 \neq 0)$$

Finally subtracting E from D we obtain.

$$\alpha_3(\lambda_2-\lambda_3)=0$$

$$\Rightarrow \alpha_3=0 \quad \because \lambda_2-\lambda_3\neq 0.$$

From D it follows that $\alpha_2=0$

From A it follows that $\alpha_1=0$

Hence $\alpha_1=\alpha_2=\alpha_3=0$. The vectors v_1, v_2, v_3 are l.i.

2 $$P_1=\left\{\begin{pmatrix}1\\0\\1\\1\end{pmatrix}+t_1\begin{pmatrix}3\\4\\6\\8\end{pmatrix}+t_2\begin{pmatrix}0\\9\\4\\0\end{pmatrix} : t_1,t_2\in\mathbb{R}\right\}$$

$$P_2=\left\{\begin{pmatrix}-2\\3\\-1\\5\end{pmatrix}+t_3\begin{pmatrix}0\\0\\-3\\-7\end{pmatrix}+t_4\begin{pmatrix}0\\0\\0\\5\end{pmatrix} : t_3,t_4\in\mathbb{R}\right\}$$

Note that t_1, t_2, t_3, t_4 are dummy variables. To find the intersection of P_1 and P_2 we need to find $t_1, t_2, t_3, t_4 \in \mathbb{R}$:

$$1 + 3t_1 = -2 \Rightarrow t_1 = -1$$

$$4t_1 + 9t_2 = 3 \Rightarrow t_2 = 7/9$$

$$1 + 6t_1 + 4t_2 = -1 - 3t_3$$

$$1 + 8t_1 = 5 - 7t_3 + 5t_4 \star$$

$$\begin{aligned} 3t_3 &= -1 - 1 - 6t_1 - 4t_2 \\ &= -2 - 6(-1) - 4 \cdot \frac{7}{9} \\ &= 4 - 4 \cdot \frac{7}{9} \\ &= 4\left(2/9\right) = 8/9 \end{aligned}$$

$$t_3 = 8/27 = .2963$$

Finally from $\star$ we obtain

$$5t_4 = 1 + 8t_1 - 5 + 7t_3$$
$$= -4 + 8(-1) + 7\cdot\frac{8}{27}$$
$$= -12 + \frac{56}{27}$$
$$= \frac{56 - 12.27}{27}$$
$$= \frac{56 - 324}{27} = -\frac{268}{27}$$
$$t_4 = -\frac{268}{135}$$
$$= -1\cdot 9852$$

$t_1 = -1$, $t_2 = \frac{7}{9} = \cdot 7778$

$t_3 = 8/27 = \cdot 2963$

$t_4 = -\frac{268}{135} = -1\cdot 9852$

3

a $\ell_1 : \begin{pmatrix} 1 \\ t_1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \boxed{x=1, z=1}$

$\ell_2 : \begin{pmatrix} 1+t_2 \\ 3+t_2 \\ -1+t_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Rightarrow x-1 = y-3 = z+1 = t_2$

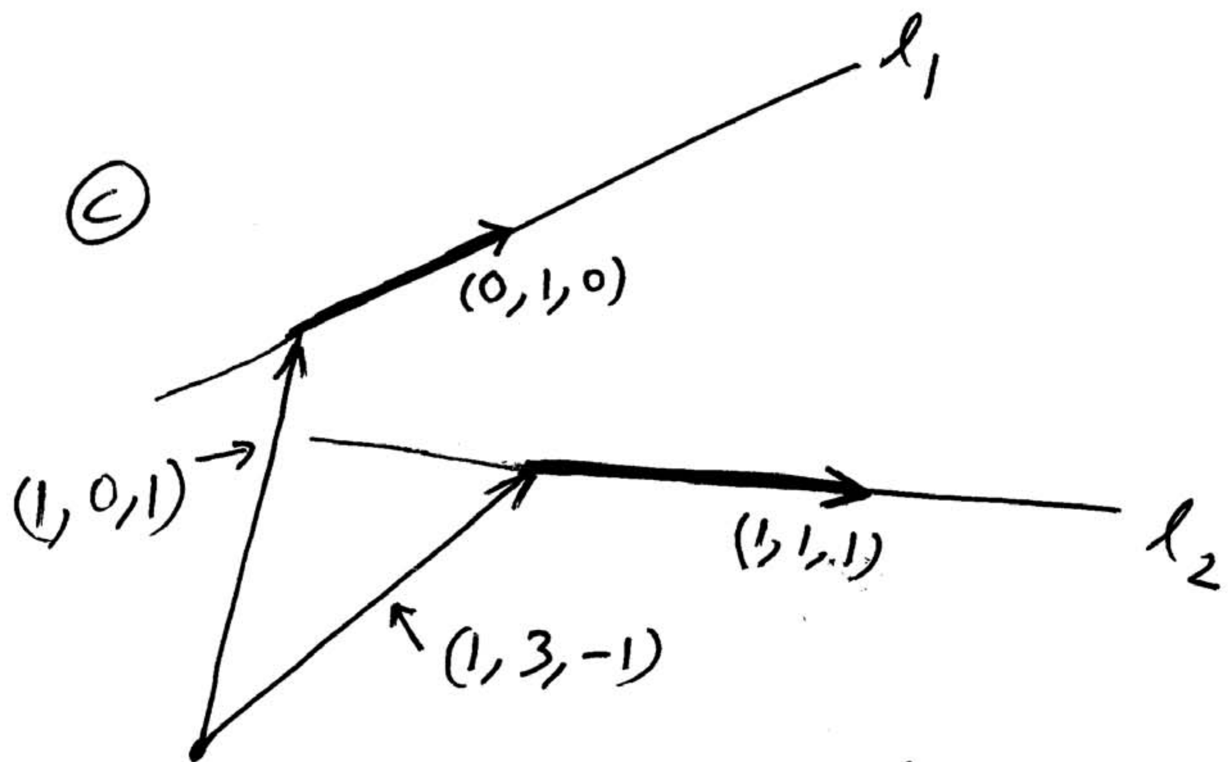
$\boxed{x-1 = y-3 = z+1}$ $\star$

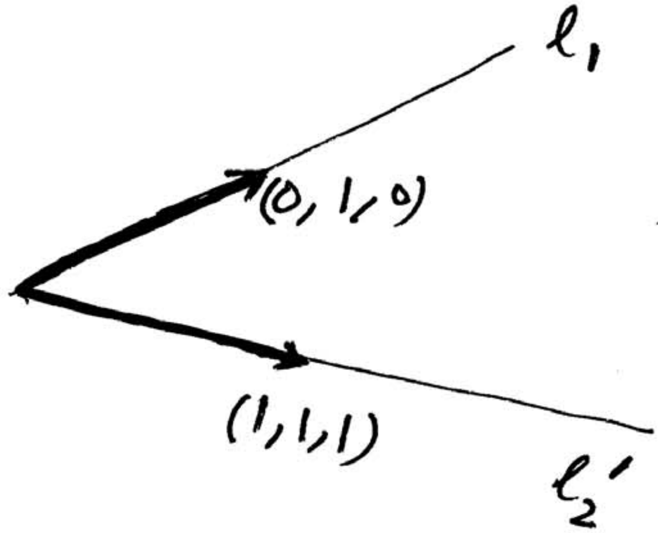
b Substituting $x=1, z=1$ in $\star$ we obtain

$1-1 = y-3 = 1+1$

which is absurd. Hence ℓ_1 & ℓ_2 do not intersect.

(c)





Since l_2' is parallel to l_2, orientation of l_2' is along the vector $(1,1,1)$

Define $u_1 = (0,1,0)$

$$u_2 = (1,1,1)$$

Let us calculate $u_1 \times u_2$

$u_1 \times u_2 = (1 \quad 0 \quad -1)$.

Equation of the plane that contains l_1 & l_2' is

$$x - z = d$$

and this plane passes through the point $(1, 0, 1)$. Hence $d = 0$

$$\boxed{x - z = 0}$$

4 $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x = -w, y = z \right\}$

Substituting $x = -w$ & $y = z$ we obtain

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -w \\ z \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} w + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} z \qquad \bigstar$$

Basis of V is

$$\left\{ \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{v_2} \right\}$$

Why? Because v_1 & v_2 span V (follows from $\bigstar$)

Also v_1 & v_2 are independent because

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\Rightarrow \left. \begin{aligned} -\alpha_1 &= 0 \\ \alpha_2 &= 0 \\ \alpha_2 &= 0 \\ \alpha_1 &= 0 \end{aligned} \right\} \Rightarrow \alpha_1 = \alpha_2 = 0 .$$

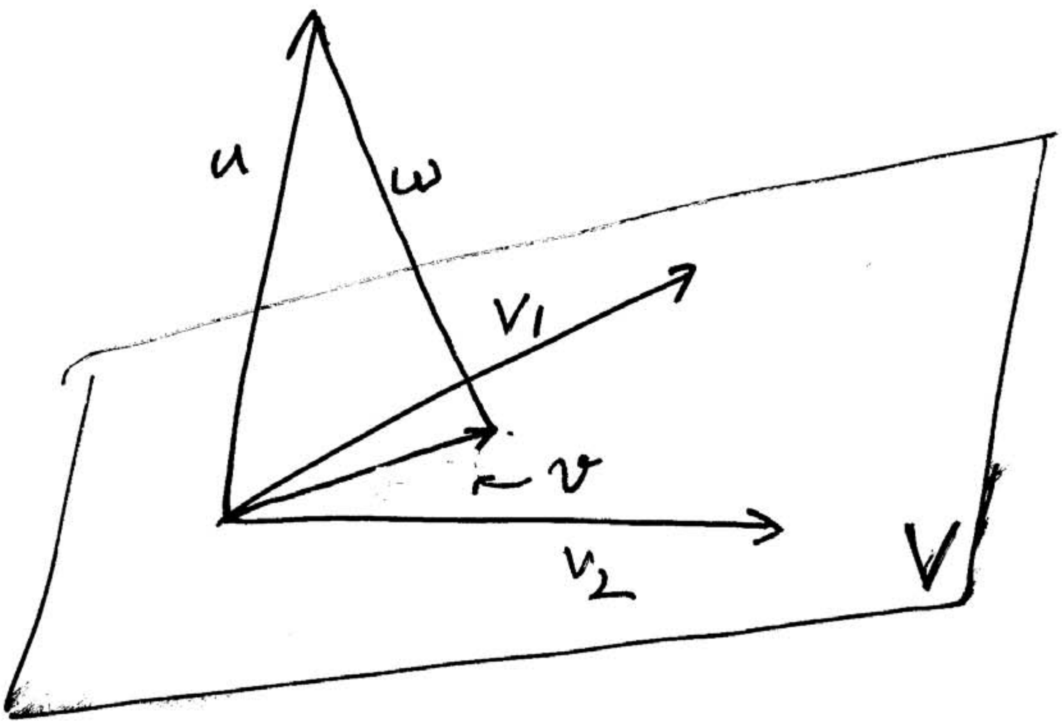
9

(a) Dimension of V =

$\#$ elements in a basis of V

$= 2$

(b)



$u = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, $v = \alpha v_1 + \beta v_2 = \text{proj}_V u$

$w = u - \text{proj}_V u$

$w \perp V \Rightarrow w \perp v_1 \;\&\; w \perp v_2$

$$w \perp v_1 \Rightarrow (u - \mathrm{proj}_V u) \cdot v_1 = 0$$

$$w \perp v_2 \Rightarrow (u - \mathrm{proj}_V u) \cdot v_2 = 0$$

$$\Downarrow$$

$$\underbrace{(\alpha v_1 + \beta v_2)}_{\mathrm{proj}_V u} \cdot v_1 = u \cdot v_1$$

$$\overbrace{(\alpha v_1 + \beta v_2)} \cdot v_2 = u \cdot v_2$$

$$\Rightarrow (v_1 \cdot v_1)\alpha + (v_2 \cdot v_1)\beta = u \cdot v_1$$

$$(v_1 \cdot v_2)\alpha + (v_2 \cdot v_2)\beta = u \cdot v_2$$

$v_1 \cdot v_1 = 2 \qquad v_1 \cdot v_2 = 0 \qquad u \cdot v_2 = 0$

$v_2 \cdot v_2 = 2 \qquad u \cdot v_1 = 0$

$\Rightarrow 2\alpha = 0 \qquad 2\beta = 0$

$\Rightarrow \alpha = \beta = 0 \qquad v = \mathrm{proj}_V u = 0v_1 + 0v_2$

$= 0$