ESE 501, Final Examination, Fall 2006

There are six questions in all of equal weights. The test is open books and notes and you can use calculators for the four arithmetic operations. I am expecting you to spend no more than 4 hours. The answers are due on 20th December, 2006. Please put your answers in the box in front of the ESE office or send it to me by e-mail: ghosh at netra.wustl.edu

DO NOT DISCUSS ANSWERS AMONG EACH OTHER BEFORE 20^{th} DEC., 2006.

GOOD LUCK

1. Let S be a subset of $\mathbb{R}^3$ described as follows:

$$S = \{(x, y, z) : x + y + z = 0\}$$

   (a) Show that S is a vector space.

   (b) Calculate a basis of S and compute its dimension.

   (c) Find the vector in S which is closest to the vector $(1, 3, -5)$ in $\mathbb{R}^3$.

2. Let A be the matrix defined as follows:

$$A = \begin{pmatrix} 6 & -1 & 5 & 7 \\ 2 & 1 & 3 & 1 \\ 8 & -3 & 5 & 11 \end{pmatrix}$$

   Let T be a linear transformation given by

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longmapsto A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

   (a) Calculate the dimension of the range and null space of T.

   (b) Explain with a clear argument if T is 1-1.

   (c) Explain with a clear argument if T is onto.

   (d) Describe all vectors that are in the preimage of $(4, 4, 2)$.

3. Let A be the matrix defined as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{pmatrix}$$

   (a) Calculate the characteristic polynomial of A.

   (b) Show that A has repeated eigenvalues at -1 repeating four times.

   (c) Calculate a set of four linearly independent eigenvectors/generalized eigenvectors of A and write down a matrix P such that $P^{-1}AP$ is in the jordan canonical form

$$J = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

   (d) Calculate J^8.
   Can you write down an expression for J^N for any positive integer N?

4. Assume that X is a state vector in $\mathbb{R}^2$ i.e.

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Let us define the matrix A as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 21 & 4 \end{pmatrix}$$

and consider the differential equation

$$\dot{X} = AX.$$

(a) Computing the eigenvalues and eigenvectors of A, sketch the phase diagram on $\mathbb{R}^2$.

(b) Calculate $X(t)$ assuming the initial condition:

$$X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) Calculate the impulse response of the following:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = 21x_1 + 4x_2 + 5u(t), \quad y(t) = x_1(t).$$

Hint: Impulse response is the response $y(t)$ assuming $x_1(0) = x_2(0) = 0$ and $u(t)$ is the unit impulse function at $t = 0$.

5. Consider a sequence of real numbers given by

$$\{x_1, x_2, x_3, x_4, \dots\}$$

where we are given that

$$x_1 = x_2 = 1$$

and the integers x_k satisfy the following recursion:

$$x_{k+2} + 0.3x_{k+1} + 0.02x_k = 0.$$

(a) Write down an expression for x_{100}.

(b) Argue what happens to x_N as N tends to infinity.

(c) If we modify the above recursion to

$$x_{k+2} + 0.3x_{k+1} + 0.02x_k = u_k.$$

where we assume that $u_1 = 1$ and $u_k = 0$ for $k = 2, 3, 4, \dots$. Continue to assume that $x_1 = x_2 = 1$. Write down an expression for x_{100}.

Hint: In this problem it might help to know that

$$\lambda^2 + 0.3\lambda + 0.02 = (\lambda + 0.1)(\lambda + 0.2)$$

6. Consider the polynomial

$$p(\lambda) = \lambda^5 + 10\lambda^4 + 40\lambda^3 + 80\lambda^2 + 80\lambda + 32$$

(a) Using Routh Hurwitz criterion, verify if $p(\lambda)$ has all roots with negative real parts.

(b) Using Jury's test, verify if $p(\lambda)$ has all roots with magnitude < 1.

(c) Describe how you would verify if $p(\lambda)$ has all roots with real part < -5.
Your description in part (c) should clearly indicate how you would calculate without actually showing me the calculation.