Solutions to Midterm Examination II

1 Ans.

$$\det(\lambda I - A) =$$

$$\det \begin{pmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & -1 & \lambda \end{pmatrix}$$

$$= \lambda \underbrace{\det \begin{pmatrix} \lambda & 0 & -1 \\ 0 & \lambda & 0 \\ -1 & -1 & \lambda \end{pmatrix}}_{\text{I}} - 1 \underbrace{\det \begin{pmatrix} 0 & \lambda & -1 \\ -1 & 0 & 0 \\ 0 & -1 & \lambda \end{pmatrix}}_{\text{II}}$$

$$\text{I} = \lambda \begin{vmatrix} \lambda & 0 \\ -1 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & \lambda \\ -1 & -1 \end{vmatrix} = \lambda^3 - \lambda$$

$$= \lambda(\lambda^2 - 1)$$

$$\text{II} = -(-1) \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda^2 - 1)$$

It follows that

$$\det(\lambda I - A) =$$

$$\lambda\left[\lambda(\lambda^2-1)\right] - 1\left[\lambda^2-1\right]$$

$$= \lambda^2(\lambda^2-1) - (\lambda^2-1) = (\lambda^2-1)(\lambda^2-1)$$

$$= (\lambda^2-1)^2 = \lambda^4 - 2\lambda^2 + 1$$

— x —

2 Ans: Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ be the eigenvector.

It follows that $Av = +1\,v$

$\Rightarrow v_3 = v_1,\ v_1 + v_2 = v_2,\ v_1 = v_3,\ v_2 + v_3 = v_4$

$\Rightarrow v_1 = v_3 = 0;\ v_2 = v_4.$

Choosing $v_2 = v_4 = 1$, we have $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = v$ is

an eigenvector.

Let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ be the generalized eigenvector.

It follows that

$$Au = u + v$$

$$\Rightarrow \begin{array}{l} u_3 = u_1 . \\ u_1 + u_2 = u_2 + 1 \\ u_1 = u_3 \\ u_2 + u_3 = u_4 + 1 \end{array} \quad \Rightarrow \begin{array}{l} u_1 = u_3 = 1 \\ u_2 = u_4 \end{array}$$

Choosing $u_2 = u_4 = 0$ we have

$u = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector.

3 Ans:

$$A^3 e^{At} = \alpha_0 I + \alpha_1 A \quad (*)$$

Since -4 is repeating twice as an eigenvalue of A, we write the following equation from (*)

$$f(\lambda) = \alpha_0 + \alpha_1 \lambda \quad (A)$$

where

$$f(\lambda) = \lambda^3 e^{\lambda t}.$$

The parameters α_0 and α_1 are computed by solving

$$f(-4) = \alpha_0 - 4\alpha_1 \leftarrow \text{substituting } \lambda = -4 \text{ in (A)}$$

$$f'(-4) = 0 + \alpha_1 \leftarrow \text{Taking derivative w.r.t } \lambda \text{ and then substituting } \lambda = -4.$$

$$f'(\lambda) = \lambda^3 t e^{\lambda t} + 3\lambda^2 e^{\lambda t}$$

$$= \lambda^2 e^{\lambda t}(3 + \lambda t)$$

$$f'(-4) = 16 e^{-4t}(3 - 4t).$$

Thus we have

$$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{bmatrix} -64 \\ 16(3-4t) \end{bmatrix} e^{-4t}$$

$a = d = 1,\ c = 0,\ b = -4$

$f = -64e^{-4t}$

$g = 16(3-4t)e^{-4t}$

$\boxed{\text{Row space computation}}$

4 Ans: switching the rows of matrix A we get

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis of row space $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

Null space computation

Null space is described as.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x = -z, \; y = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} z$$

Basis of null space $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

5 **Ans:**

Let $v = (a \quad b \quad c \quad d)$ be the required vector.

$\because v \in P$ we have

$$3a + 2b + c - 2d = 0$$

$$2a + 3b + 2c - d = 0$$

$\because v \perp u$ we have

$$a - b + c + d = 0$$

The 3 equations are written as

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 3 & 2 & 1 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Row Reduction:

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 3 & 2 & 1 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 5 & -2 & -5 \\ 0 & 5 & 0 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 5 & -5 & 5 & 5 \\ 0 & 5 & -2 & -5 \\ 0 & 5 & 0 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 5 & 0 & 3 & 0 \\ 0 & 5 & -2 & -5 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 5 & 0 & 3 & 0 \\ 0 & 5 & 0 & -3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 10 & 0 & 6 & 0 \\ 0 & 5 & 0 & -3 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 10 & 0 & 0 & -6 \\ 0 & 5 & 0 & -3 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

The three equations are equivalently written as

$$\left.\begin{aligned} 10a - 6d &= 0 \\ 5b - 3d &= 0 \\ 6c + 6d &= 0 \end{aligned}\right\} \quad \begin{aligned} a &= \tfrac{3}{5} d. \\ b &= 3/5\, d. \\ c &= -d. \end{aligned}$$

Choosing $d=5$, the vector v is given by

$$v=\begin{pmatrix}3\\3\\-5\\5\end{pmatrix}$$

6 Ans:

Choose $u_1=\begin{pmatrix}1\\1\end{pmatrix}$, $u_2=\begin{pmatrix}1\\-1\end{pmatrix}$

$$Wu_1=\lambda u_1$$

$$\Rightarrow \left.\begin{array}{l}\cos\theta+\sin\theta=\lambda\\ \sin\theta+\cos\theta=\lambda\end{array}\right\} \Rightarrow \boxed{\lambda=\cos\theta+\sin\theta}$$

$$Wu_2=\lambda u_2$$

$$\Rightarrow \left.\begin{array}{l}\cos\theta-\sin\theta=\lambda\\ \sin\theta-\cos\theta=-\lambda\end{array}\right\} \Rightarrow \boxed{\lambda=\cos\theta-\sin\theta}$$

The two eigenvalues

The normalized eigenvectors are given by

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

The columns of P are the normalized eigenvectors.