Homework Nine

1. Let

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$$

be a linear transformation defined as follows

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 - x_1 \\ 2x_1 + x_2 - x_3 \end{pmatrix}$$

   (a) Write down the range and null space of T as span of a set of linear independent vectors.

   (b) Obtain the cartesian equation of the range and null space.

   (c) Choose a point $p = (3 \ \ 5 \ \ 2 \ \ 1)^T$ in the range of T. Describe the pre image of T, i.e. find

$$S = \{x \in \mathbb{R}^3 : T(x) = p\}.$$

   (d) Is the space S (in part (c)) a vector space? If not, what is it?

   (e) Let P be a plane in $\mathbb{R}^3$ described by the equation

$$x_1 - x_3 = 0.$$

      i. Calculate the image of P under the transformation T, call it T(P).

      ii. Is T(P) a vector space, justify with a reason. If not, what is it?

2. Let

$$A = \begin{pmatrix} 2 & 5 & 9 \\ 3 & 2 & 6 \\ 1 & -3 & 8 \end{pmatrix},$$

and let b be a 3×1 vector with the property that b, Ab, A^2b are linearly independent, so that

$$B = \{b, Ab, A^2b\}$$

form a basis of $\mathbb{R}^3$. Define a linear transformation

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

where

$$A^{n-1}b \longmapsto A^n b, n = 1, 2, 3, 4, \ldots.$$

   (a) If a vector v has coordinates $(1, 2, 5)$ with respect to basis B, i.e.

$$v = b + 2Ab + 5A^2b$$

   find coordinates of $T(v)$ with respect to the same basis B.

   (b) If a vector v has coordinates (α, β, γ) with respect to basis B, i.e.

$$v = \alpha b + \beta Ab + \gamma A^2 b$$

   find coordinates of $T(v)$ with respect to the same basis B.

3. Let $P_3(t)$ be the set of all polynomials in t of degree ≤ 3. Define a linear transformation

$$T : P_3(t) \rightarrow P_3(t)$$

where

$$p(t) \longmapsto \frac{d}{dt}p(t).$$

(a) Calculate the range and null space of T.

(b) Let

$$B = \{1, 1+t, t-t^2, t^3\}$$

be a basis of $P_3(t)$. If a polynomial $p(t)$ has coordinates $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ with respect to the basis B. Calculate the coordinates $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ of $T(p(t))$ with respect to B.

(c) Write down a matrix M such that

$$M \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix}$$

for all possible choices of coordinates α_1, β_1, γ_1 and δ_1.

4. Let $P_3(t)$ be the set of all polynomials in t of degree ≤ 3 and let $P_2(t)$ be the set of all polynomials in t of degree ≤ 2. Define a linear transformation

$$T : P_2(t) \to P_3(t)$$

where

$$p(t) \longmapsto \int p(t)\, dt.$$

(a) Calculate the range and null space of T.

(b) Let

$$B_1 = \{1, 1+t, t-t^2, t^3\}$$

be a basis of $P_3(t)$ and let

$$B_2 = \{1, 1+t, t-t^2\}$$

be a basis of $P_2(t)$. If a polynomial $p(t)$ has coordinates $(\alpha_1, \beta_1, \gamma_1)$ with respect to the basis B_2. Calculate the coordinates $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ of $T(p(t))$ with respect to B_1.

(c) Write down a matrix M such that

$$M \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix}$$

for all possible choices of coordinates α_1, β_1 and γ_1.

5. Consider the ordinary differential equation

$$\dot{x} = Ax + bu, \quad y = c^T x$$

where $x(0) = 0$, $c^T = (1 \;\; 0)$,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution $y(t)$ can be written as

$$y(t) = \int_0^t c^T e^{A(t-\tau)} b u(\tau) d\tau.$$

(a) Assume $u(t) = \alpha + \beta t$ for α, $\beta \in \mathbb{R}$, calculate $y(t)$ in terms of α and β and show that it is a polynomial of degree ≤ 3, i.e.

$$y(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3.$$

(b) Define a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$ as follows

$$T : \mathbb{R}^2 \to \mathbb{R}^4$$

where

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Calculate the range and the null space of T.

6. Let us define

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Recall that an ordinary differential equation

$$\dot{x} = Ax + bu, x(0) = 0$$

has the following solution

$$x(t) = \int_0^t e^{A(t-\tau)} bu(\tau) d\tau.$$

At $t = 1$, it follows that

$$x(1) = \int_0^1 e^{A(1-\tau)} bu(\tau) d\tau,$$

where $x(1) \in \mathbb{R}^2$, and $u(\tau)$ is a function defined in the interval $[0 \ \ 1]$.

(a) Calculate $x(1)$ for $u(\tau) = 1$, $\tau \in [0 \ \ 1]$.

(b) Calculate $x(1)$ for $u(\tau) = \tau$, $\tau \in [0 \ \ 1]$.

(c) Let c_1 and c_2 be two arbitrary real constants, we define

$$u(\tau) = b^T e^{-A^T \tau} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and write $x(1)$ as

$$x(1) = e^A \left[\int_0^1 e^{-A\tau} b b^T e^{-A^T \tau} d\tau \right] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

i. Calculate the 2×2 matrix

$$M = \left[\int_0^1 e^{-A\tau} b b^T e^{-A^T \tau} d\tau \right]$$

and check its rank. Is it 2 ?

ii. Calculate the 2×2 matrix

$$M = \left[\int_0^1 e^{-A\tau} b b^T e^{-A^T \tau} d\tau \right]$$

taking the matrices A and b as

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and check its rank. Is it 2 or a 1 ?

iii. Choose

$$x(1) = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

and taking M as in part (i), calculate

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} e^{-A} x(1)$$

and hence calculate

$$u(\tau) = b^T e^{-A^T \tau} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Remark: the constructed $u(\tau)$ is the forcing function (also called the control) that drives the state from $x(0) = (0\ \ 0)^T$ to $x(1) = (7\ \ 9)^T$

7. Let us define

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix},\ c^T = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

and consider an ordinary differential equation of the form

$$\dot{x} = Ax, y = c^T x, x(0) = x_0.$$

Recall that $y(t)$ is given as follows

$$y(t) = c^T e^{At} x_0.$$

(a) Calculate $y(t)$ assuming

$$x_0 = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

(b) We would now like to calculate x_0 given $y(t)$. This is done as follows:

$$y(t) = c^T e^{At} x_0$$

implies that

$$c y(t) = c c^T e^{At} x_0$$

which further implies that

$$e^{A^T t} c y(t) = e^{A^T t} c\, c^T e^{At} x_0.$$

Integrating both sides with respect to t in the interval $[0\ \ 1]$ we obtain

$$\int_0^1 e^{A^T t} c y(t) dt = \left[\int_0^1 e^{A^T t} c\, c^T e^{At} dt \right] x_0.$$

i. Calculate the 2×2 matrix N given by

$$N = \left[\int_0^1 e^{A^T t} c\, c^T e^{At} dt \right]$$

and check its rank. Is it 2?

ii. Choose $y(t)$ in part (a) and calculate

$$\xi = \int_0^1 e^{A^T t} c y(t) dt.$$

iii. Writing $\xi = N x_0$, calculate $x_0 = N^{-1} \xi$.

The matrices M and N in the problems 6 and 7 are called 'Controllability' and 'Observability' Gramians respectively.