SSM 501, Home Work Seven

1. (a) Manufacture a 4×4 matrix A with eigenvalues at 5, 7, 13 and 17 and eigenvectors precisely at

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ -2 \end{pmatrix}, v_4 = \begin{pmatrix} 3 \\ 4 \\ -5 \\ -1 \end{pmatrix}. \tag{1}$$

   (b) Orthogonalize the vectors v_1, v_2, v_3, v_4 using Gramm-Schmidt Orthogonalization. Call the orthogonal vectors u_1, u_2, u_3, u_4.

   (c) Manufacture a 4×4 symmetric, p;ositive definite matrix B with eigenvalues at 5, 7, 13 and 17 and eigenvectors precisely at u_1, u_2, u_3, u_4.

   (d) Consider the ellipsoid in $\mathbb{R}^4$ given by

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 100 \tag{2}$$

   Write down the homogeneous, quadratic, polynomial equation of the ellipsoid using variables x_1, x_2, x_3, x_4. Define a new set of variables y_1, y_2, y_3, y_4 such that the ellipsoid is given as

$$5y_1^2 + 7y_2^2 + 13y_3^2 + 17y_4^2 = 100$$

   (e) Compute the point of intersection p between the line

$$\ell = \{x_1 = 3t, x_2 = 4t, x_3 = -5t, x_4 = -t : t \in \mathbb{R}\}$$

   and the ellipsoid (2).

   (f) Compute equation of the tangent plane to the ellipsoid at the point p. (Excluded from HW, will discuss in class)

   (g) Consider a function

$$\phi(x_1, x_2, x_3, x_4) = 8x_1 + 9x_2 + 7x_3 - 12x_4.$$

   Find maximum and minimum value of ϕ subject to the constraint that (x_1, x_2, x_3, x_4) belongs to the ellipsoid. (Excluded from HW, will discuss in class)

2. Let us define

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

   (a) Construct a 4×4 matrix A and a vector of initial conditions X_0 such that when you solve

$$\dot{X} = AX, \; X(0) = X_0$$

   we get $x_1 = \sin 2t, \; x_3 = \sin 4t$. Also calculate x_2 and x_4 for your choice of A and X_0.

   (b) Construct a 4×4 matrix A and a vector of initial conditions X_0 such that when you solve

$$\dot{X} = AX, \; X(0) = X_0$$

   we get $x_1 = te^{-3t}\sin 7t$. Also calculate x_2, x_3 and x_4 for your choice of A and X_0.

3. Consider the following 2^{nd} order o.d.e.

$$m\ddot{x} + b\dot{x} + kx = \sin 4t$$

assuming initial conditions $x(0) = 5$ and $\dot{x}(0) = 15$. Take $m = 1$, and $k = 1$. Solve the equation for the following three different values of b.

   (a) $b = 1$ (underdamped case)
   (b) $b = 3$ (overdamped case)
   (c) $b = 2$ (critically damped case).

   Using any plotting software, plot $x(t)$ as a function of t.

4. Determine the type and stability of the critical point. Find a real general solution. Sketch or plot some trajectories in the phase plane. (Pick any 5 for credit, it may be too much to do all)

   (a)
   $$\dot{x_1} = x_1, \dot{x_2} = 2x_2;$$
   (b)
   $$\dot{x_1} = 2x_1 + x_2, \dot{x_2} = 5x_1 - 2x_2;$$
   (c)
   $$\dot{x_1} = x_1 + 2x_2, \dot{x_2} = 2x_1 + x_2;$$
   (d)
   $$\dot{x_1} = -6x_1 - x_2, \dot{x_2} = -9x_1 - 6x_2;$$
   (e)
   $$\dot{x_1} = -2x_1 + 2x_2, \dot{x_2} = -2x_1 - 2x_2;$$
   (f)
   $$\dot{x_1} = x_1 - 2x_2, \dot{x_2} = 5x_1 - x_2;$$
   (g)
   $$\dot{x_1} = x_2, \dot{x_2} = -9x_1;$$
   (h)
   $$\dot{x_1} = -x_1 + 4x_2, \dot{x_2} = 3x_1 - 2x_2;$$
   (i)
   $$\dot{x_1} = -2x_1 - 6x_2, \dot{x_2} = -8x_1 - 4x_2;$$
   (j)
   $$\dot{x_1} = -x_1, \dot{x_2} = -5x_1 - x_2;$$
   (k)
   $$\dot{x_1} = 2x_1 + x_2, \dot{x_2} = 6x_1 + 2x_2;$$