ESE 501, Home Work 5

1. Let v_1 and v_2 be two linearly independent vectors in R^n such that $\|v_1\| = 1$ and $\|v_2\| = 1$, but the two vectors are not necessarily orthogonal. Let us denote by $P = [v_1 \ v_2]$ the homogeneous 2-plane in R^n spanned by the vectors v_1 and v_2. Let us assume that θ is the angle between the two vectors v_1 and v_2.

   (a) Let u be any vector that does not belong to the plane P, show that

   $$\text{proj}_P u = \frac{\text{proj}_{v_1} u + \text{proj}_{v_2} u - \cos\theta\,[(u.v_2)v_1 + (u.v_1)v_2]}{\sin^2\theta}$$

   (b) Additionally if v_1 and v_2 are orthogonal vectors, i.e. if $\theta = 90$ conclude that

   $$\text{proj}_P u = \text{proj}_{[v_1 \ v_2]} u = \text{proj}_{v_1} u + \text{proj}_{v_2} u.$$

   This exercise illustrates the advantage of an orthonormal set of basis vectors.

2. Let $v_1, v_2, \cdots, v_m$ be a set of m linearly independent vectors in R^n, $m \leq n$. For any vector u in R^n, show that

   (a) $\text{proj}_{[v_1, v_2, \cdots, v_m]}\text{u}=$

   $$\begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix} \begin{pmatrix} v_1.v_1 & v_2.v_1 & \cdots & v_m.v_1 \\ v_1.v_2 & v_2.v_2 & \cdots & v_m.v_2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ v_1.v_m & v_2.v_m & \cdots & v_m.v_m \end{pmatrix}^{-1} \begin{pmatrix} u.v_1 \\ u.v_2 \\ \cdots \\ \cdots \\ u.v_m \end{pmatrix}.$$

   (b) If $v_1, v_2, \cdots, v_m$ is an orthonormal set of vectors in R^n, show that
   $\text{proj}_{[v_1, v_2, \cdots, v_m]}\text{u}=$

   $$\begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix} \begin{pmatrix} u.v_1 \\ u.v_2 \\ \cdots \\ \cdots \\ u.v_m \end{pmatrix}$$

   where $u.v_i$ is the i^{th} coordinate of the projection.
   Remark: When the basis vectors are not orthonormal, the i^{th} coordinate depends upon all the basis vectors. This example illustrates the advantage of choosing orthonormal vectors.

   (c) Take $m = 3$, $n = 5$ and define the vectors as follows

   $$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 7 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 8 \\ -6 \end{pmatrix} \text{ and } u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

   Calculate $\text{proj}_{[v_1, v_2, v_3]}\text{u}$

(3) Let us assume

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 & -1 \\ 3 & 6 & 2 & 8 & 0 \\ 2 & 4 & -1 & 3 & 1 \end{pmatrix}$$

(a) Write down the cartesian equation of the row space, column space and the null space of A.

(b) Obtain a basis of the spaces in (a). Hence write down rank and nullity of A and verify that rank + nullity $= 5$

(c) Orthogonalize the basis vectors obtained in (b) by G.S. orthogonalization procedure

(d) Project the vector $(0\ 0\ 0\ 0\ 1)$ on the null space of A.

(4) Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Show by explicit computation that

$$\det(\lambda I + A) = \lambda^3 + \Delta_1 \lambda^2 + \Delta_2 \lambda + \Delta_1$$

where

$\Delta_1 =$ trace A, $\Delta_3 = \det A$.

Can you also find Δ_2 ??

5. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Write down the characteristic polynomial of A and verify Cayley Hamilton Theorem.

6. Let

$$A = \begin{pmatrix} a & b \\ c & D \end{pmatrix}$$

where $a \neq 0$ is a scalar
c is a $n \times 1$ vector
b is a $1 \times n$ vector
D is a $n \times n$ matrix

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Show if it is true that.

$$\det A = (\det D \cdot a - b\,\mathrm{adj}\,D\,c)$$

Verify the above formula (if it is true) choosing

$$A = \left(\begin{array}{c|cc} 1 & 3 & 9 \\ \hline 4 & 5 & 7 \\ 5 & 8 & 17 \end{array}\right)$$