Solutions to Home Work 9

1 a

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 - x_1 \\ 2x_1 + x_2 - x_3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}}_{v_1} x_1 + \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{v_2} x_2 + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}}_{v_3} x_3$$

Check v_1, v_2, v_3 for linear Independence

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

$$\Rightarrow \left.\begin{array}{r} \alpha + \beta = 0 \\ \beta + \gamma = 0 \\ -\alpha + \gamma = 0 \\ 2\alpha + \beta - \gamma = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \alpha = -\beta \\ \beta = -\gamma \\ \alpha = \gamma \\ \gamma = 2\alpha + \beta \end{array}\right\} \Rightarrow \alpha = -\beta = \gamma$$

$\alpha = 1, \beta = -1, \gamma = 1$ is a sol$^{\underline{n}}$

$\therefore v_1, v_2, v_3$ are linearly dependent.

Range of T

$$\mathcal{R}(T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\quad\;\; v_1 \qquad\quad v_2$$

We have dropped v_3.
Check easily that v_1 & v_2
are linearly independent.

Null space of T

$$\mathcal{N}(T) = \{ x : T(x) = 0 \}.$$

$$= \{ (\alpha, \beta, \gamma) : \alpha = -\beta = \gamma \}.$$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

"from the computation
in the previous page."

(b) Null space is a line in $\mathbb{R}^3$, homogeneous and passing through the point $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

To find equation of such a line l, we need to find 2 vectors in $\mathbb{R}^3$ that are

- linearly independent
- perpendicular to $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

If (a, b, c) is one such vector, it follows that

$$a - b + c = 0$$

$$\Rightarrow a = b - c$$

If $b = 1, c = 0$ then $a = 1$

If $b = 0, c = 1$ then $a = -1$

$\therefore \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are a required pair of l.i. vectors.

Equation of ℓ is

$$\left.\begin{aligned} x+y+0z&=0 \\ -x+0y+z&=0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x&=-y \\ x&=z \end{aligned}}$$

— x —

Range space is a 2-plane in $\mathbb{R}^4$, which is homogeneous and passes through the vectors

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

To find equation of this plane P, we need to find 2 vectors in $\mathbb{R}^4$ that are

- linearly independent.
- perpendicular to $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

If $(a\ b\ c\ d)$ is one such vector, it follows that

$$(a\ b\ c\ d)\begin{pmatrix}1\\0\\-1\\2\end{pmatrix}=0$$

$$(a\ b\ c\ d)\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=0$$

$$\Downarrow$$

$$\begin{aligned}a-c+2d&=0\\a+b+d&=0\end{aligned}\quad\Rightarrow\quad\begin{aligned}a-c+2d&=0\\b+d+c-2d&=0\end{aligned}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}a&=c-2d\\b&=d-c.\end{aligned}}$$

If $c=1, d=0$ then $a=1, b=-1$

If $c=0, d=1$ then $a=-2, b=1$

Hence

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$ are two such vectors. ~~that~~ are clearly l.i.

In the co-ordinates (x, y, z, w) the equation of the plane is given by

$$\boxed{\begin{aligned} x - y + z &= 0 \\ -2x + y + w &= 0 \end{aligned}}$$

c $\quad p = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 1 \end{pmatrix}$

Need to find $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 1 \end{pmatrix}$$

One choice of (x_1, x_2, x_3) is obtained by choosing $x_3 = 0$ and writing

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} x_2 = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 1 \end{pmatrix}$$

$x_1 + x_2 = 3$

$x_2 = 5$ $\quad \Rightarrow x_1 = -2,\ x_2 = 5,\ x_3 = 0$

$x_1 = -2$

$2x_1 + x_2 = 1$

$\begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$ is one pt. in the preimage

To find the preimage of p we add the null space to any one point in the preimage. This gives us

$$\underbrace{\begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{\text{This is an affine line in } \mathbb{R}^3}$$

comes from null space

To find equation of the affine line in $\mathbb{R}^3$ we write

$$\begin{aligned} x &= -2+t \\ y &= 5-t \\ z &= t \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} x &= -2+z \\ y &= 5-z \end{aligned}}$$

Equation of the affine line in $\mathbb{R}^3$.

(d) It is not a vector space because it does not contain the origin $(0,0,0)$.

In fact it is also not "closed" under addition and scalar multiplication,

ie if
v_1, v_2 are two points on the affine line
$v_1 + v_2$ does not belong to the line

and
αv_1 does not belong to the line either.

(e) Let us first parameterize the plane P as follows:

Find two l.i. vectors in P

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$ would be a choice.

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{u_1} + t_2 \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{u_2} \right\}$$

It follows that

$$T(P) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t_1 T(u_1) + t_2 T(u_2) \right\}$$

$$T(u_1) = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix}, \quad T(u_2) = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

Hence $T(P) =$

$$\operatorname{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix} \right\}$$

Note that they are linearly independent.

Since $T(P)$ is a span of 2 vectors it is a vector space. Since the 2 vectors are l.i.

$\dim T(P) = 2$.

— × —

2 Ans:

$$A = \begin{pmatrix} 2 & 5 & 9 \\ 3 & 2 & 6 \\ 1 & -3 & 8 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda-2 & -5 & -9 \\ -3 & \lambda-2 & -6 \\ -1 & 3 & \lambda-8 \end{pmatrix}$$

$\det(\lambda I - A) =$

$$(\lambda-2)\begin{vmatrix} \lambda-2 & -6 \\ 3 & \lambda-8 \end{vmatrix} + 5\begin{vmatrix} -3 & -6 \\ -1 & \lambda-8 \end{vmatrix} - 9\begin{vmatrix} -3 & \lambda-2 \\ -1 & 3 \end{vmatrix}$$

$$= (\lambda - 2)\left[\lambda^2 - 10\lambda + 16 + 18\right]$$
$$+ 5\left[-3\lambda + 24 - 6\right]$$
$$- 9\left[-9 + \lambda - 2\right]$$

$$= (\lambda - 2)\left[\lambda^2 - 10\lambda + 34\right]$$
$$+ \left[-15\lambda + 90\right]$$
$$- 9\left[\lambda - 11\right]$$

$$= \lambda^3 - 10\lambda^2 + 34\lambda$$
$$- 2\lambda^2 + 20\lambda - 68$$
$$- 15\lambda + 90$$
$$- 9\lambda + 99$$

$$= \lambda^3 - 12\lambda^2 + 54\lambda - 24\lambda + 189 - 68$$
$$= \lambda^3 - 12\lambda^2 + 30\lambda + 121$$

9

$$v = b + 2\,Ab + 5\,A^2 b$$

$$T(v) = T(b) + 2\,T(Ab) + 5\,T(A^2 b)$$

$$= Ab + 2\,A^2 b + 5\,A^3 b$$

But

$$A^3 - 12\,A^2 + 30\,A + 121\,I = 0$$

$$\Rightarrow A^3 = 12\,A^2 - 30\,A - 121\,I$$

$$\therefore T(v) = Ab + 2\,A^2 b + 60\,A^2 b - 150\,Ab - 605\,b$$

$$= -605\,b - 149\,Ab + 62\,A^2 b$$

co-ordinates of $T(v)$ are

$$(-605, -149, 62)$$

(b) If

$$v = \alpha b + \beta Ab + \gamma A^2 b$$

Then

$$T(v) = \alpha Ab + \beta A^2 b + \gamma A^3 b$$

$$\gamma A^3 b = 12\gamma A^2 b - 30\gamma Ab - 121\gamma b$$

$$\therefore T(v) = (-121\gamma) b$$
$$+ (\alpha - 30\gamma) Ab$$
$$+ (\beta + 12\gamma) A^2 b$$

$\therefore$ co-ordinates of $T(v)$ are

$$(-121\gamma, \alpha - 30\gamma, \beta + 12\gamma)$$

3 Ans:

An element of $P_3(t)$ is given by

$$\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 = p(t).$$

$$T(p(t)) = 0 + \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2.$$

$$= \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2.$$

a Clearly $T(p(t))$ is any polynomial of degree ≤ 2, which is the range of T.

Null space of T is obtained by setting

$T(p(t)) =$ zero polynomial

$\Rightarrow \alpha_1 = 0,\ 2\alpha_2 = 0,\ 3\alpha_3 = 0$

$\therefore$ Null space of T is the set of constant polynomials of $P_3(t)$.

(b)

$$p(t) = \alpha_1 + \beta_1(1+t) + \gamma_1(t-t^2) + \delta_1(t^3)$$
$$= (\alpha_1+\beta_1) + (\beta_1+\gamma_1)t - \gamma_1 t^2 + \delta_1 t^3$$

$$T(p(t)) = (\beta_1+\gamma_1) - 2\gamma_1 t + 3\delta_1 t^2$$

Let (a, b, c, d) be the co-ordinates of $T(p)$ w.r.t. the basis $\mathcal{B}$. It follows that

$$a + b(1+t) + c(t-t^2) + d t^3 = (\beta_1+\gamma_1) - 2\gamma_1 t + 3\delta_1 t^2$$

$$\Rightarrow \left.\begin{aligned} d &= 0 \\ a+b &= \beta_1+\gamma_1 . \\ b+c &= -2\gamma_1 \\ c &= -3\delta_1 \end{aligned}\right\} \quad \begin{aligned} c &= -3\delta_1 \\ b &= 3\delta_1 - 2\gamma_1 . \\ a &= \beta_1+\gamma_1 - 3\delta_1 + 2\gamma_1 . \\ &= \beta_1 - 3\delta_1 + 3\gamma_1 . \\ d &= 0 \end{aligned}$$

$(\beta_1 - 3\delta_1 + 3\gamma_1, 3\delta_1 - 2\gamma_1, -3\delta_1, 0)$ are the co-ordinates.

$\therefore$

$$\alpha_2 = \beta_1 - 3\delta_1 + 3\gamma_1 .$$

$$\beta_2 = 3\delta_1 - 2\gamma_1 .$$

$$\gamma_2 = -3\delta_1 .$$

$$\delta_2 = 0 .$$

$$\Downarrow .$$

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\shortparallel \atop M} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix}$$

4 Ans.

Let $q(t) \in P_3(t)$

$p(t) \in P_2(t)$

$$p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$$

$$T(p(t)) = \int p(t)\,dt$$

$$= \alpha_0 t + \frac{\alpha_1 t^2}{2} + \alpha_2 \frac{t^3}{3}.$$

$$= t\left(\alpha_0 + \frac{\alpha_1}{2} t + \frac{\alpha_2}{3} t^2\right)$$

(a) Range of T are polynomials $q(t)$ in $P_3(t)$ that are of the form

$$q(t) = t \times (\text{polynomial of degree} \leq 2)$$

These are all polynomials in $P_3(t)$ that have a zero at $t=0$. and are of the form

$* t + * t^2 + * t^3.$

To compute null space of T, we write

$$T(p(t)) = 0 \text{ (zero polynomial)}.$$

$$\Rightarrow \alpha_0 = 0, \ \frac{\alpha_1}{2} = 0, \ \frac{\alpha_2}{3} = 0$$

$$\Rightarrow \alpha_0 = \alpha_1 = \alpha_2 = 0.$$

Thus the null space is trivial ie consists of only the zero polynomial.

(b) $p(t) = \alpha_1 + \beta_1 (1+t) + \gamma_1 (t - t^2).$

$$= (\alpha_1 + \beta_1) + (\beta_1 + \gamma_1) t - \gamma_1 t^2$$

$$T(p(t)) = (\alpha_1 + \beta_1) t + (\beta_1 + \gamma_1) \frac{t^2}{2} - \gamma_1 \frac{t^3}{3}.$$

writing $T(p(t))$ as

$$\alpha_2 + \beta_2(1+t) + \gamma_2(t-t^2) + \delta_2 t^3$$
$$= (\alpha_2+\beta_2) + (\beta_2+\gamma_2)t - \gamma_2 t^2 + \delta_2 t^3.$$

we have

$$\left.\begin{aligned} \alpha_2+\beta_2 &= 0 \\ \beta_2+\gamma_2 &= \alpha_1+\beta_1 \\ -\gamma_2 &= \frac{\beta_1+\gamma_1}{2}. \\ \delta_2 &= -\gamma_1/3. \end{aligned}\right\}.$$

$$\Downarrow$$

$$\delta_2 = -\frac{1}{3}\gamma_1.$$

$$\gamma_2 = -\frac{1}{2}\beta_1 - \frac{1}{2}\gamma_1.$$

$$\beta_2 = \alpha_1 + \beta_1 + \frac{1}{2}\beta_1 + \frac{1}{2}\gamma_1.$$
$$= \alpha_1 + \frac{3}{2}\beta_1 + \frac{1}{2}\gamma_1.$$

$$\alpha_2 = -\beta_2$$
$$= -\alpha_1 - \frac{3}{2}\beta_1 - \frac{1}{2}\gamma_1$$

co-ordinates of $T(p(t))$ are

$$\begin{pmatrix} -\alpha_1 - \frac{3}{2}\beta_1 - \frac{1}{2}\gamma_1 \\ \alpha_1 + \frac{3}{2}\beta_1 + \frac{1}{2}\gamma_1 \\ -\frac{1}{2}\beta_1 - \frac{1}{2}\gamma_1 \\ -\frac{1}{3}\gamma_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix}$$

$$\|$$

(c)

$$\underbrace{\begin{pmatrix} -1 & -\frac{3}{2} & -\frac{1}{2} \\ 1 & 3/2 & 1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/3 \end{pmatrix}}_{M} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$$

5 Ans:

writing $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ we have

$$\dot{x}_1 = x_2 . \qquad x_1(0) = x_2(0) = 0 .$$
$$\dot{x}_2 = u$$

$$y = x_1 \quad , u = \alpha + \beta t .$$

$$\dot{x}_2 = u \Rightarrow x_2(t) = \alpha t + \beta \frac{t^2}{2} + \overset{=0}{x_2(0)}$$

Hence $x_2(t) = \alpha t + \frac{\beta}{2} t^2$

$$\dot{x}_1 = x_2 \Rightarrow x_1(t) = \alpha \frac{t^2}{2} + \frac{\beta}{2} \frac{t^3}{3}$$
$$= \frac{\alpha}{2} t^2 + \frac{\beta}{6} t^3 + \overset{=0}{x_1(0)}$$

$$\therefore \boxed{x_1(t) = \frac{\alpha}{2} t^2 + \frac{\beta}{6} t^3} .$$

$$y(t) = 0 + 0t + \frac{\alpha}{2} t^2 + \frac{\beta}{6} t^3 .$$

Hence $\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = \frac{\alpha}{2}, \alpha_3 = \frac{\beta}{6}$.

b

$$\begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 0\\ 0 & 0\\ 1/2 & 0\\ 0 & 1/6\end{pmatrix}}_{= N}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\therefore T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto N\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Range of T is the

$$\operatorname{span}\left\{\begin{pmatrix}0\\0\\1/2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1/6\end{pmatrix}\right\}$$

$$= \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

Range of T is the set of all points in $\mathbb{R}^4$ whose first two co-ordinates are zero.

$$R(T) = \left\{ \begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix} : x=0,\ y=0 \right\}$$

Null space of T is given by all $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$:

$$N \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \alpha = \beta = 0 .$$

Null space of T is therefore trivial.

— x —

6 Ans:

a $$x(1) = \int_0^1 e^{A(1-\tau)} b\, u(\tau)\, d\tau .$$

Since $u(\tau) = 1$ we have

$$x(1) = \int_0^1 e^{A(1-\tau)} b\, d\tau .$$

b As in a

$$x(1) = \int_0^1 e^{A(1-\tau)} b\, \tau\, d\tau .$$

Let us first calculate e^{At}.

$e^{At} = ??$

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}$$

Eigenvalues at $\lambda = -2$, $\lambda = -3$.

$$e^{At} = \alpha_0 I + \alpha_1 A.$$

$$\left.\begin{aligned} e^{-2t} &= \alpha_0 - 2\alpha_1 \\ e^{-3t} &= \alpha_0 - 3\alpha_1. \end{aligned}\right\} \Rightarrow e^{-2t} - e^{-3t} = \alpha_1$$

$$\boxed{\begin{aligned} \alpha_1 &= e^{-2t} - e^{-3t}. \\ 3e^{-2t} - 2e^{-3t} &= \alpha_0 \end{aligned}}$$

$$e^{At} = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} + \begin{pmatrix} -2\alpha_1 & \alpha_1 \\ 0 & -3\alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 - 2\alpha_1 & \alpha_1 \\ 0 & \alpha_0 - 3\alpha_1 \end{pmatrix}$$

$$e^{At}b = \begin{pmatrix} \alpha_1 \\ \alpha_0 - 3\alpha_1 \end{pmatrix}$$

$$\boxed{\alpha_0 - 3\alpha_1 = e^{-3t}.}$$

$$e^{At}b = \begin{pmatrix} e^{-2t} - e^{-3t} \\ e^{-3t} \end{pmatrix}$$

Ans to part (a)

$$x(1) = \begin{bmatrix} \int_0^1 e^{-2(1-\tau)} - e^{-3(1-\tau)} \, d\tau \\ \int_0^1 e^{-3(1-\tau)} \, d\tau. \end{bmatrix}$$

Ans to part (b)

$$x(1) = \begin{bmatrix} \int_0^1 e^{-2(1-\tau)}\tau - e^{-3(1-\tau)}\tau \, d\tau. \\ \int_0^1 e^{-3(1-\tau)}\tau \, d\tau. \end{bmatrix}$$

(c)

$$e^{-A\tau} b = \begin{pmatrix} e^{+2\tau} - e^{+3\tau} \\ e^{+3\tau} \end{pmatrix}$$

$$e^{-A\tau} b b^T e^{-A^T\tau}.$$

$$= (e^{-A\tau} b)(e^{-A\tau} b)^T$$

$$= \begin{pmatrix} e^{2\tau} - e^{3\tau} \\ e^{3\tau} \end{pmatrix} \begin{pmatrix} e^{2\tau} - e^{3\tau} & e^{3\tau} \end{pmatrix}$$

$$= \begin{pmatrix} e^{4\tau} + e^{6\tau} - 2e^{5\tau}. & e^{5\tau} - e^{6\tau} \\ e^{5\tau} - e^{6\tau} & e^{6\tau}. \end{pmatrix}$$

$$= W(\tau).$$

$$M = \int_0^1 W(\tau)\, d\tau.$$

$$= \begin{bmatrix} \frac{e^{4\tau}}{4} + \frac{e^{6\tau}}{6} - \frac{2}{5}e^{5\tau}. & \frac{e^{5\tau}}{5} - \frac{e^{6\tau}}{6}. \\ \frac{e^{5\tau}}{5} - \frac{e^{6\tau}}{6}. & \frac{e^{6\tau}}{6} \end{bmatrix}_{\tau=0,1.}$$

(i) $$M = \begin{pmatrix} \frac{e^4}{4} + \frac{e^6}{6} - \frac{2}{5}e^5 & \frac{e^5}{5} - \frac{e^6}{6} \\ \frac{e^5}{5} - \frac{e^6}{6} & \frac{e^6}{6}. \end{pmatrix}$$

$$- \begin{pmatrix} \frac{1}{4} + \frac{1}{6} - \frac{2}{5} & \frac{1}{5} - \frac{1}{6} \\ \frac{1}{5} - \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

Check using matlab that

det $M \neq 0$

Hence M is of rank 2.

(ii) $A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$

$$e^{-A\tau} = \begin{pmatrix} e^{2\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix}$$

$$e^{-A\tau} b = \begin{pmatrix} 0 \\ e^{\tau} \end{pmatrix}$$

$$e^{-A\tau} b\, b^T e^{-A^T\tau}$$

$$= (e^{-A\tau} b)(e^{-A\tau} b)^T$$

$$= \begin{pmatrix} 0 \\ e^{\tau} \end{pmatrix} \begin{pmatrix} 0 & e^{\tau} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & e^{2\tau} \end{pmatrix} = W(\tau).$$

$$M = \int_0^1 W(\tau)\, d\tau = \begin{pmatrix} 0 & 0 \\ 0 & \frac{e^{2\tau}}{2} \end{pmatrix} \Bigg|_{\tau=0}^{1}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \frac{e^2}{2} - \frac{1}{2} \end{pmatrix}$$

$\boxed{\text{M is of rank } \underline{1}}$

iii

$$x(1) = e^{A} M \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} e^{-A} \begin{pmatrix} 7 \\ 9 \end{pmatrix} \leftarrow$$ use matlab to compute this.

$$b^T e^{-A^T \tau} = \begin{pmatrix} e^{2\tau} - e^{3\tau} & e^{3\tau} \end{pmatrix}$$

$$u(\tau) = c_1 (e^{2\tau} - e^{3\tau}) + c_2 e^{3\tau}.$$

$$\boxed{u(\tau) = c_1 e^{2\tau} + (c_2 - c_1) e^{3\tau}}$$

7 Ans.

From 6 it follows that

$$c^T e^{At} =$$

$$(\alpha_0 - 2\alpha_1 \quad \alpha_1)$$

$$\alpha_0 - 2\alpha_1 = e^{-2t}.$$

$$\therefore c^T e^{At} =$$

$$(e^{-2t} \quad e^{-2t} - e^{-3t})$$

(a) $\therefore y(t) = 5e^{-2t} + 7(e^{-2t} - e^{-3t}).$

$$= 12e^{-2t} - 7e^{-3t}.$$

(b)

(i)

$$e^{A^T t} c c^T e^{At}$$

$$= (c^T e^{At})^T (c^T e^{At})$$

$$= \begin{pmatrix} e^{-2t} \\ e^{-2t} - e^{-3t} \end{pmatrix} \begin{pmatrix} e^{-2t} & e^{-2t} - e^{-3t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-4t} & e^{-4t} - e^{-5t} \\ e^{-4t} - e^{-5t} & e^{-4t} + e^{-6t} - 2e^{-5t} \end{pmatrix}$$

$$\shortparallel$$

$$Q(\tau).$$

$$N = \int_0^1 Q(\tau)\, d\tau.$$

$$\left.\begin{pmatrix} \dfrac{e^{-4t}}{-4} & \dfrac{e^{-4t}}{-4} - \dfrac{e^{-5t}}{-5} \\ \dfrac{e^{-4t}}{-4} - \dfrac{e^{-5t}}{-5} & \dfrac{e^{-4t}}{-4} + \dfrac{e^{-6t}}{-6} - 2\dfrac{e^{-5t}}{-5} \end{pmatrix}\right|_{t=0}^{t=1}$$

Use matlab to calculate N.

and check that rank $N = 2$

$\det N \neq 0$.

ii

$$\xi = \begin{bmatrix} \int_0^1 e^{-2t} y(t)\, dt \\ \int_0^1 (e^{-2t} - e^{-3t})\, y(t)\, dt \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^1 e^{-2t} (12e^{-2t} - 7e^{-3t})\, dt \\ \int_0^1 (e^{-2t} - e^{-3t})(12e^{-2t} - 7e^{-3t})\, dt \end{bmatrix}$$

iii $x_0 = N^{-1} \xi$.

This part is left out because it requires matlab.