HW-9 (Answers).

(1) $$x(t) = \underbrace{e^{At}}_{f_1} \underbrace{\int_0^t e^{-A\tau} b u(\tau) d\tau}_{f_2}.$$

$$\dot{x} = f_1 \frac{df_2}{dt} + \frac{df_1}{dt} f_2.$$

$$f_1 \frac{df_2}{dt} = e^{At}\left[e^{-At} b u(t)\right] = b u(t).$$

$$\frac{df_1}{dt} f_2 = A e^{At} f_2 = A f_1 f_2 = A x(t).$$

$$\therefore \boxed{\dot{x} = Ax + bu}$$

(2) Let

$$B = P^{-1} A P$$

where B is in the (Jordan) canonical form i.e.

$$B = \begin{pmatrix} \lambda_1 & 0 \text{ or } 1 & & 0 \\ & \lambda_2 & 1 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$\det B = \lambda_1 \cdots \lambda_n$$

$$\operatorname{trace} B = \sum \lambda_i$$

$$\left.\begin{aligned} \det B &= \det A \\ \operatorname{trace} B &= \operatorname{trace} A \end{aligned}\right\} \text{ why??}$$

Because char. polynomial of A and B are same.

(3) Ans:

Let $\left.\begin{array}{l}\lambda_1, v_1 \\ \lambda_2, v_2 \\ \lambda_3, v_3\end{array}\right\}$ be the eigenvalue eigenvector pairs.

$A v_1 = \lambda_1 v_1$.

$A v_2 = \lambda_2 v_2$.

$A v_3 = \lambda_3 v_3$.

(a) $B \begin{pmatrix} v_i \\ 0 \end{pmatrix} = \begin{pmatrix} A v_i \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_i v_i \\ 0 \end{pmatrix} = \lambda_i \begin{pmatrix} v_i \\ 0 \end{pmatrix}$.

Hence $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of B with $\begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} v_3 \\ 0 \end{pmatrix}$ the corresponding eigenvectors.

(b)

Suppose not, ie assume that

$\exists \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ another eigenvector of B,

l.i. with respect to $\begin{pmatrix} v_i \\ 0 \end{pmatrix}$, $i=1,2,3$.

Eigenvector of B are at either $\lambda_1, \lambda_2, \lambda_3$.

without any loss of generality assume

that $\lambda_1, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is an eigenvalue eigenvector pair.

$$B \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\Rightarrow A\xi_1 + \xi_2 = \lambda_1 \xi_1.$$

$$A\xi_2 = \lambda_1 \xi_2.$$

$$A \xi_2 = \lambda_1 \xi_2$$

$\Rightarrow \xi_2 = \mu v_1$ ie ξ_2 must be an eigenvector of A.

Thus

$$A \xi_1 + \mu v_1 = \lambda_1 \xi_1 \quad (*)$$

writing

$$\xi_1 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

we have

$$A \xi_1 = \alpha_1 A v_1 + \alpha_2 A v_2 + \alpha_3 A v_3 .$$

$$= \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \alpha_3 \lambda_3 v_3 .$$

from $(*)$ we have

$$\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \alpha_3 \lambda_3 v_3 + \mu v_1 =$$

$$\lambda_1 \alpha_1 v_1 + \lambda_1 \alpha_2 v_2 + \lambda_1 \alpha_3 v_3 .$$

comparing co-efficients of v_1 we obtain

$$\alpha_1 \lambda_1 + \mu = \alpha_1 \lambda_1 \Rightarrow \mu = 0 .$$

Thus $\xi_2 = 0$. which violates the assumption that

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} v_3 \\ 0 \end{pmatrix} \text{ are l.i.}$$

$\begin{pmatrix} v_i \\ 0 \end{pmatrix}, \begin{pmatrix} v_i \\ v_i \end{pmatrix}$ is a pair of eigenvector gen-eigenvector of B.

$$B \begin{pmatrix} v_i \\ 0 \end{pmatrix} = \lambda_i \begin{pmatrix} v_i \\ 0 \end{pmatrix}.$$

$$B \begin{pmatrix} v_i \\ v_i \end{pmatrix} = \begin{pmatrix} Av_i + v_i \\ Av_i \end{pmatrix} = \begin{pmatrix} \lambda_i v_i + v_i \\ \lambda_i v_i \end{pmatrix}$$

$$= \lambda_i \begin{pmatrix} v_i \\ v_i \end{pmatrix} + \begin{pmatrix} v_i \\ 0 \end{pmatrix}.$$

(4) $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$P^{-1} A P = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

(a) $P^{-1} e^{At} P = \begin{pmatrix} e^{-2t} & te^{-2t} & t^2/2\, e^{-2t} \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{pmatrix}$

(b) $A = P \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} P^{-1}$.

5 Ans:

$$S = I + 2A + 3A^2 + \cdots + NA^{N-1}$$

$$SA = A + 2A^2 + 3A^3 + \cdots + NA^N$$

$$S - SA =$$

$$I + A + A^2 + A^3 + \cdots + A^{N-1} - NA^N$$

$$S_1 = I + A + A^2 + \cdots + A^{N-1}$$

$$S_1 A = A + A^2 + \cdots + A^N$$

$$S_1 (I - A) = I - A^N$$

$$S_1 = (I - A^N)(I - A)^{-1}$$

$$S(I - A) = (I - A^N)(I - A)^{-1} - NA^N$$

$$S = (I - A^N)(I - A)^{-2} - NA^N(I - A)^{-1}$$

$$\boxed{\lim_{N\to\infty} S = (I - A)^{-2}}$$

because eigenvalues of A are all inside the unit circle.