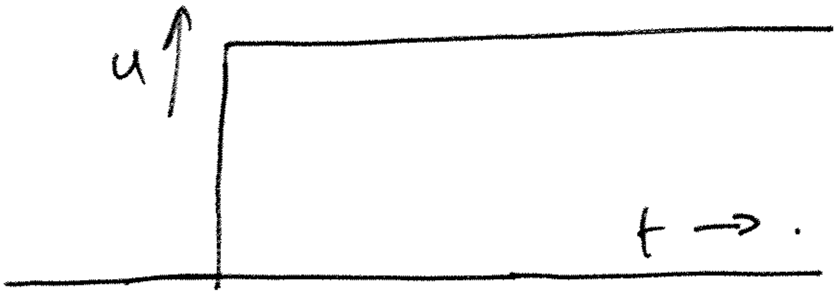
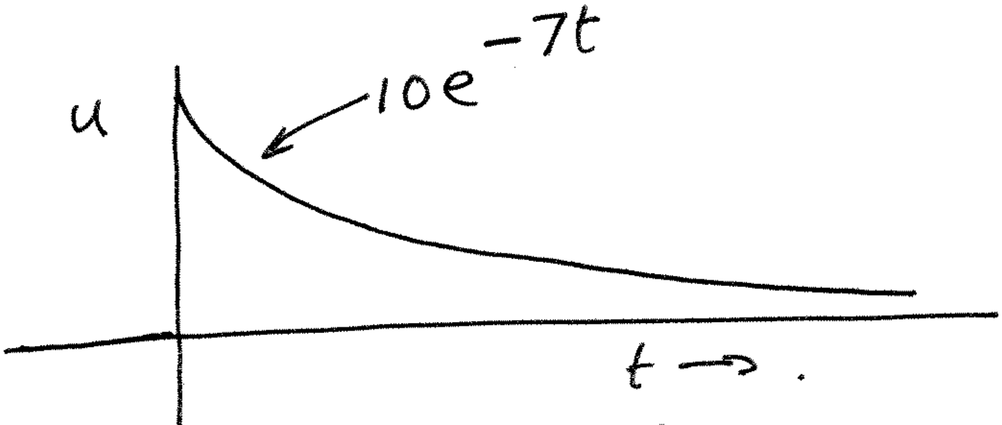
H. W. 8
Solutions.

① Ans:



when $u(t) = 10 \quad t \geq 0$

$$y(t) = 25(1 - e^{-2t}) \quad t \geq 0$$



when $u(t) = 10e^{-7t} \quad t \geq 0$

$$y(t) = 10(e^{-2t} - e^{-7t}) \quad t \geq 0$$

Define

$$H(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & 0 \le t \le t_0 \end{cases}$$

$$y(t) = 25\left(1 - e^{-2t}\right)$$
$$- 25\left(1 - e^{-2(t-5)}\right) H(t-5).$$
$$+ 10\left(e^{-2(t-5)} - e^{-7(t-5)}\right) H(t-5)$$

$$\boxed{\begin{array}{l} \text{For } t \le 5 \\ y(t) = 25\left(1 - e^{-2t}\right) \end{array}}$$

For $t \geq 5$

$$y(t) = 25\left(1 - e^{-2t}\right)$$
$$- 25\left(1 - e^{-2(t-5)}\right)$$
$$+ 10\left(e^{-2(t-5)} - e^{-7(t-5)}\right)$$

$$= -25e^{-2t} + 25e^{-2(t-5)} + 10e^{-2(t-5)} - 10e^{-7(t-5)}$$

$$= -25e^{-2t} + 35e^{-2(t-5)} - 10e^{-7(t-5)}.$$

② Ans:

$$y(t) = \int_0^t e^{-2(t-\tau)} 5.10 \sin 15\tau \, d\tau .$$

$$= 50 e^{-2t} \int_0^t e^{2\tau} \sin 15\tau \, d\tau .$$

$$\int e^{2t} \sin 15t \, dt =$$

$$\frac{e^{2t}}{229} \left(2 \sin 15t - 15 \cos 15t\right)$$

$$y(t) = 50 e^{-2t} \left[\frac{e^{2t}}{229} \left(2 \sin 15t - 15 \cos 15t\right) + \frac{1}{229} 15 \right]$$

$$= \frac{50.15}{229} e^{-2t} + \underbrace{\frac{50}{229} \left[2 \sin 15t - 15 \cos 15t\right]}_{y_{ss}} .$$

③ Ans:

$x_1 = y$

$x_2 = \dot{y}$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$h(t) = c e^{At} b$$

$$= \frac{1}{3} \sqrt{3} \, e^{-t} \sin(\sqrt{3}\, t)$$

$$= \frac{1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}\, t)$$

(b) $y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau.$

$$= \int_0^t h(\tau)\, u(t-\tau)\, d\tau.$$

Since $u(t)$ is unit step we have

$$u(t-\tau) = 1 \qquad 0 \le \tau \le t.$$

$$\therefore\ y(t) = \int_0^t h(\tau)\, d\tau.$$

$$= \frac{1}{\sqrt{3}} \int_0^t e^{-\tau} \sin(\sqrt{3}\,\tau)\, d\tau.$$

$$\int e^{-\tau} \sin(\sqrt{3}\,\tau)\, d\tau$$

$$= \frac{e^{-\tau}}{4} \left(-\sin\sqrt{3}\,\tau - \sqrt{3}\cos\sqrt{3}\,\tau\right)$$

$$y(t) = \frac{1}{\sqrt{3}} \left[\frac{e^{-t}}{4} \left(-\sin\sqrt{3}\,t - \sqrt{3}\cos\sqrt{3}\,t\right) + \frac{1}{4}\left(\sqrt{3}\right) \right]$$

$$y(t) = \frac{1}{4} - \frac{1}{4\sqrt{3}} e^{-t} \left(\sin \sqrt{3}\, t + \sqrt{3} \cos \sqrt{3}\, t \right)$$

④ Ans:

$$\dddot{y} - 2\dot{y} - 4y = u(t)$$

ⓐ $\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = \dddot{y} = 2\dot{y} + 4y + u$

$= 2x_2 + 4x_1 + u$

$\dot{\underline{x}} = A\underline{x} + bu$

$y = c\underline{x}$

$c = (1 \quad 0 \quad 0)$; $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix}$

ⓑ characteristic poly =

$$\lambda^3 - 0\lambda^2 - 2\lambda - 4$$

Eigenvalues $2, -1+i, -1-i$

(c)

$$\begin{array}{cccc} 0 & 4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ & 1 & -2 & 0 \end{array}$$

$\Delta_1 = 0$

$\Delta_2 = 0 - 4 = -4$

$\Delta_3 = 4 \cdot (-4) = -16$

Eigenvalues are not all in the left half plane.

(d) $h(t) =$

$$-\frac{3}{10} e^{-t} \sin t - \frac{1}{10} e^{-t} \cos t + \frac{1}{10} e^{2t}$$

(e) $y(t) = \int_0^t h(\tau)\, d\tau$

$y(t)$ is unbounded.

$$\int_0^t e^{-\tau} \sin\tau \, d\tau =$$

$$\frac{e^{-\tau}}{2}\left[-\sin\tau - \cos\tau\right]\Big|_0^t$$

$$= -\frac{e^{-t}}{2}(\sin t + \cos t) + \frac{1}{2}$$

$$= \frac{1}{2}\left[1 - e^{-t}(\sin t + \cos t)\right]$$

$$\int_0^t e^{-\tau} \cos\tau \, d\tau =$$

$$\frac{e^{-\tau}}{2}\left(-\cos\tau + \sin\tau\right)\Big|_0^t$$

$$= \frac{e^{-t}}{2}\left(\sin t - \cos t\right) + \frac{1}{2}$$

$$= \frac{1}{2}\left(1 + e^{-t}(\sin t - \cos t)\right)$$

$$\int_0^t e^{2\tau} \, d\tau = \frac{e^{2\tau}}{2}\Big|_0^t = \frac{1}{2}\left(e^{2t} - 1\right)$$

$$y(t) = \frac{1}{20}\left(e^{2t} - 1\right)$$

$$- \frac{1}{20}\left[1 + e^{-t}(\sin t - \cos t)\right]$$

$$- \frac{3}{20}\left[1 - e^{-t}(\sin t + \cos t)\right]$$

$$\dddot{y} + 4\ddot{y} + 6\dot{y} + 4y = u(t).$$

(a) $c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

$b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{pmatrix}$$

(b) char poly =

$$\lambda^3 + 4\lambda^2 + 6\lambda + 4.$$

Eigenvalues $-2, -1+i, -1-i$.

(c) $\begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 0 \\ 0 & 4 & 4 \end{pmatrix} \leftarrow$ R.H. Criterion

$\Delta_1 = 4$

$\Delta_2 = 24 - 4 = 20$

$\Delta_3 = 4 \cdot 20 = 80.$

(d)

$h(t) =$

$\frac{1}{2} e^{-t} \sin t$

$- \frac{1}{2} e^{-t} \cos t \cdot$

$+ \frac{1}{2} e^{-2t} \cdot$

$$\int_0^t e^{-2\tau} d\tau$$

$$= \left. \frac{e^{-2\tau}}{-2} \right|_0^t$$

$$= -\frac{1}{2} \left(e^{-2t} - 1 \right)$$

$$= \frac{1}{2} \left(1 - e^{-2t} \right).$$

(e) $y(t) = \int_0^t h(\tau) d\tau.$

$$= \frac{1}{4} \left[1 - e^{-t} (\sin t + \cos t) \right]$$

$$- \frac{1}{4} \left[1 + e^{-t} (\sin t - \cos t) \right]$$

$$+ \frac{1}{4} \cdot \left[1 - e^{-2t} \right].$$

(5)

(a) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & -\frac{3}{4} & -\frac{3}{2} \end{pmatrix}$ $\quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(b) $p(\lambda) = \lambda^3 + \frac{3}{2}\lambda^2 + \frac{3}{4}\lambda + \frac{1}{8}$

$= \left(\lambda + \frac{1}{2}\right)^3$

(c) $q(\lambda) = (\lambda-1)^3 \, p\left(\frac{\lambda+1}{\lambda-1}\right)$

$= (\lambda+1)^3 + \frac{3}{2}(\lambda+1)^2(\lambda-1) + \frac{3}{4}(\lambda+1)(\lambda-1)^2$
$+ \frac{1}{8}(\lambda-1)^3$

$= \frac{27}{8}\left(\lambda^3 + \lambda^2 + \frac{1}{3}\lambda + \frac{1}{27}\right)$

Routh Hurwitz on.

$$\lambda^3+\lambda^2+\frac{1}{3}\lambda+\frac{1}{27}$$

$$\begin{pmatrix} 1 & 1/27 & 0 \\ 1 & 1/3 & 0 \\ 0 & 1 & 1/27 \end{pmatrix}$$

$$\Delta_1 = 1 > 0$$

$$\Delta_2 = \frac{1}{3} - \frac{1}{27} = \frac{8}{27} > 0$$

$$\Delta_3 = \frac{1}{27}\cdot\frac{8}{27} > 0$$

A has all roots inside the unit.

circle.

(d) Let λ be the eigenvalue of A.

$$A^k = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$\therefore \lambda^k = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$$

$$k\lambda^{k-1} = \alpha_1 + 2\alpha_2 \lambda$$

$$k(k-1)\lambda^{k-2} = 2\alpha_2$$

$$\alpha_2 = \frac{k(k-1)}{2} \lambda^{k-2}.$$

$$cA^k b = \alpha_2 = \frac{k(k-1)}{2} \left(-\frac{1}{2}\right)^{k-2}.$$

$$h_k = \frac{(k-1)(k-2)}{2} \cdot \left(-\frac{1}{2}\right)^{k-3}.$$

(e) $$y(n) = cb + cAb + \cdots + cA^{n-1}b$$

$$= c(I + A + \cdots + A^{n-1})b$$

$$S = I + A + \cdots + A^{n-1}.$$

$$SA = \quad A + \quad + A^{n-1} + A^n$$

$$S(I-A) = I - A^n$$

$$S = (I - A^n)(I-A)^{-1}.$$

$$\therefore\ y(n) = c(I-A^n)(I-A)^{-1} b$$

$$y(k) = c(I-A^k)(I-A)^{-1} b$$

$$A^k = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$I - A^k = I(1-\alpha_0) - \alpha_1 A - \alpha_2 A^2.$$

$$\begin{aligned} c(I-A^k) &= (1-\alpha_0)(1 \quad 0 \quad 0) \\ &\quad -\alpha_1 (0 \quad 1 \quad 0) \\ &\quad -\alpha_2 (0 \quad 0 \quad 1) \\ &= (1-\alpha_0 \quad -\alpha_1 \quad -\alpha_2) \end{aligned}$$

$$\alpha_2 = \frac{k(k-1)}{2} \lambda^{k-2}.$$

$$\alpha_1 = k\lambda^{k-1} - k(k-1)\lambda^{k-1}.$$

$$= (k - k^2 + k)\lambda^{k-1}.$$

$$= (2k - k^2)\lambda^{k-1}.$$

$$\alpha_0 = \lambda^k - k\lambda^k + k(k-1)\lambda^k - \frac{k(k-1)}{2}\lambda^k.$$

$$= \left[1 - k + k(k-1) - \frac{k(k-1)}{2}\right]\lambda^k.$$

$$= \left[(k-1)^2 - \frac{k(k-1)}{2}\right]\lambda^k$$

$$= (k-1)\left[k-1 - \frac{k}{2}\right]\lambda^k$$

$$= (k-1)\left(\frac{k}{2} - 1\right)\lambda^k.$$

$$(I-A)^{-1} =$$

$$\begin{pmatrix} .9630 & .7407 & .2963 \\ -.0370 & .7407 & .2563 \\ -.0370 & -.2593 & .2963 \end{pmatrix}$$

$$\therefore (I-A)^{-1} b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .2963 .$$

$$Y(k) = C\,(I - A^n)(I-A)^{-1} b$$

$$= (1 - \alpha_0 - \alpha_1 - \alpha_2) .2963 .$$

$$= \left\{ 1 - \left[\frac{(k-1)(k-2)}{2} \lambda^k + k(2-k)\lambda^{k-1} + \frac{k(k-1)}{2} \lambda^{k-2} \right] \right\} .2963 .$$

Clearly $Y(k)$ is bounded.

$$y(k) = c(I - A^k)(I - A)^{-1} b$$

$$= (1 - \alpha_0 - \alpha_1 - \alpha_2) \cdot 2963$$

$$= \cdot 2963 \left\{ 1 - \left[\frac{(k-1)(k-2)}{2} \lambda^k + k(2-k)\lambda^{k-1} + \frac{k(k-1)}{2} \lambda^{k-2} \right] \right\}$$

Clearly $y(k)$ is bounded because

$\lambda = -1/2$.

Note: You can do the 2^{nd}. recursion analogously.

(5) Ans:-

(a) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{27}{8} & -\frac{27}{4} & -\frac{9}{2} \end{pmatrix}$ $\quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(b) $p(\lambda) = \lambda^3 + \frac{9}{2}\lambda^2 + \frac{27}{4}\lambda + \frac{27}{8}$

$= \left(\lambda + 3/2\right)^3$

(c) $q(\lambda) = (\lambda+1)^3 + \frac{9}{2}(\lambda+1)^2(\lambda-1)$

$+ \frac{27}{4}(\lambda+1)(\lambda-1)^2$

$+ \frac{27}{8}(\lambda-1)^3$

$$(\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$$

$$(\lambda+1)^2(\lambda-1) = (\lambda^2+2\lambda+1)(\lambda-1)$$

$$= \lambda^3 + \lambda^2 - \lambda - 1$$

$$(\lambda+1)(\lambda-1)^2 = (\lambda^2-2\lambda+1)(\lambda+1)$$

$$= \lambda^3 - \lambda^2 - \lambda + 1$$

$$(\lambda-1)^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 1$$

$$q(\lambda) = \left(1 + \frac{9}{2} + \frac{27}{4} + \frac{27}{8}\right)\lambda^3$$

$$\left(3 + \frac{9}{2} - \frac{27}{4} - \frac{81}{8}\right)\lambda^2$$

$$\left(3 - \frac{9}{2} - \frac{27}{4} + \frac{81}{8}\right)\lambda$$

$$\left(1 - \frac{9}{2} + \frac{27}{4} - \frac{27}{8}\right)1$$

$$q'(\lambda) = \lambda^3 + \frac{3 + 9/2 - \frac{27}{4} - \frac{81}{8}}{1 + 9/2 + \frac{27}{4} + \frac{27}{8}}\lambda^2 + \frac{3 - \frac{9}{2} - \frac{27}{4} + \frac{81}{8}}{x}\lambda$$

$$+ \frac{1 - \frac{9}{2} + \frac{27}{4} - \frac{27}{8}}{*}$$

$q'(\lambda) =$

$$\lambda^3 + (-.6)\lambda^2 + .12\lambda + -.008$$

$$= \lambda^3 - .6\lambda^2 + .12\lambda - .008$$

$$\begin{pmatrix} -.6 & -.008 & 0 \\ 1 & .12 & 0 \\ 0 & -.6 & -.008 \end{pmatrix}$$

Routh matrix.

Unstable

(d) write

$$A^{k-1} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$\lambda^{k-1} = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$$

$$(k-1)\lambda^{k-2} = \alpha_1 + 2\alpha_2 \lambda$$

$$(k-1)(k-2)\lambda^{k-3} = 2\alpha_2 \Rightarrow \boxed{\alpha_2 = \frac{(k-1)(k-2)}{2} \lambda^{k-3}}$$

$$2\alpha_2 \lambda = (k-1)(k-2)\lambda^{k-2}.$$

$$\alpha_1 = (k-1)\left[1-k+2\right]\lambda^{k-2}$$

$$\boxed{\alpha_1 = (k-1)(3-k)\lambda^{k-2}}$$

$$\alpha_1 \lambda = (k-1)(3-k)\lambda^{k-1}.$$

$$\alpha_2 \lambda^2 = \frac{(k-1)(k-2)}{2} \lambda^{k-1}.$$

$$\boxed{\alpha_0 = \lambda^{k-1}\left[1 - (k-1)(3-k) - \frac{(k-1)(k-2)}{2}\right]}$$

$$h_k = \alpha_0 Cb + \alpha_1 CAb + \alpha_2 CA^2 b$$

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}$$

$cA^2b = 1 \qquad cb = cAb = 0$

$$\therefore \boxed{h_k = \alpha_2 = \frac{(k-1)(k-2)}{2} \lambda^{k-3}.}$$

ⓔ $Y_1 = cb$

$Y_2 = cb + cAb$

$\vdots$

$Y_k = cb + cAb + \cdots + cA^{k-1}b.$

$= c(I + A + \cdots + A^{k-1})b$

$S = I + A + \cdots + A^{k-1}.$

$SA = \quad A + \quad + A^{k-1} + A^k$

$S(I-A) = I - A^k \Rightarrow S = (I-A^k)(I-A)^{-1}$

$$Y_k = c\,(I - A^k)\,(I - A)^{-1} b$$

writing

$$A^k = \beta_0 I + \beta_1 A + \beta_2 A^2$$

$$c\,(I - A^k) = (1 - \beta_0 \quad -\beta_1 \quad -\beta_2)$$

$$(I - A)^{-1} b = \cdot 064 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$Y_k = \cdot 064 \left[1 - (\beta_0 + \beta_1 + \beta_2) \right]$$

— x —

To calculate $\beta_0, \beta_1, \beta_2$ we have

$$\lambda^k = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2.$$

$$k \lambda^{k-1} = \beta_1 + 2\beta_2 \lambda.$$

$$k(k-1) \lambda^{k-2} = 2\beta_2.$$

$$\begin{pmatrix} 1 & \lambda & \lambda^2 \\ 0 & 1 & 2\lambda \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \lambda^k \\ k\lambda^{k-1}. \\ k(k-1)\lambda^{k-2} \end{pmatrix}$$

$$(\beta_0 + \beta_1 + \beta_2) =$$

$$(1 \quad 1 \quad 1) \begin{pmatrix} 1 & \lambda & \lambda^2 \\ 0 & 1 & 2\lambda \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda^k \\ k\lambda^{k-1}. \\ k(k-1)\lambda^{k-2} \end{pmatrix}$$

$$= (1 \quad 2.5 \quad 3.125) \begin{pmatrix} \lambda^k \\ k\lambda^{k-1} \\ k(k-1)\lambda^{k-2} \end{pmatrix}$$

$$Y_k = .064 \left[1 - \left(\lambda^k + 2.5 k \lambda^{k-1} + 3.125\, k(k-1)\lambda^{k-2} \right) \right]$$

$$\lambda = -3/2.$$