H. W-7 (Answers)

1 a Let A be the required matrix.

It follows that

$$A v_1 = v_1 \lambda_1$$
$$A v_2 = v_2 \lambda_2 \qquad \lambda_1 = 5, \lambda_2 = 7$$
$$A v_3 = v_3 \lambda_3 \qquad \lambda_3 = 13, \lambda_4 = 17.$$
$$A v_4 = v_4 \lambda_4$$

Define $P = (v_1 \ v_2 \ v_3 \ v_4)$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{pmatrix}$$

we have

$$A P = P \Lambda \Rightarrow A = P \Lambda P^{-1}.$$

```
>> P=[1 1 2 3;0 3 3 4;1 -1 5 -5;1 -1 -2 -1]

P =

     1     1     2     3
     0     3     3     4
     1    -1     5    -5
     1    -1    -2    -1

>> lambda=[5 0 0 0;0 7 0 0;0 0 13 0;0 0 0 17]

lambda =

     5     0     0     0
     0     7     0     0
     0     0    13     0
     0     0     0    17

>> A=P*lambda*inv(P)

A =

   21.6923  -10.4615   -2.7692  -13.9231
   20.3077   -6.5385   -3.2308  -17.0769
  -13.4615    8.3077   14.8462    3.6154
   -7.0000    4.0000   -0.0000   12.0000

>> A=P*lambda*round(inv(P)*det(P))

A =

       -1128         544         144         724
       -1056         340         168         888
         700        -432        -772        -188
         364        -208           0        -624

>> p=det(P)

p =

   -52

>>
```

$$A = \begin{pmatrix} \frac{1128}{52} & -\frac{544}{52} & -\frac{144}{52} & -\frac{724}{52} \\ \frac{1056}{52} & -\frac{340}{52} & -\frac{168}{52} & -\frac{888}{52} \\ -\frac{700}{52} & \frac{432}{52} & \frac{772}{52} & \frac{188}{52} \\ -\frac{364}{52} & \frac{208}{52} & 0 & \frac{624}{52} \end{pmatrix}$$

③

ⓑ $u_1 = \frac{v_1}{\|v_1\|}$

$$u_2' = v_2 - \mathrm{proj}_{u_1} v_2 \,,\quad u_2 = \frac{u_2'}{\|u_2'\|}$$

$$u_3' = v_3 - \left(\mathrm{proj}_{u_1} v_3 + \mathrm{proj}_{u_2} v_3\right),\quad u_3 = \frac{u_3'}{\|u_3'\|}$$

$$u_4' = v_4 - \left(\mathrm{proj}_{u_1} v_4 + \mathrm{proj}_{u_2} v_4 + \mathrm{proj}_{u_3} v_4\right)$$

$$u_4 = \frac{u_4'}{\|u_4'\|}$$

The Gram-Schmidt Orthogonalization.

```
>> v1=[1 0 1 1];
>> v2=[1 3 -1 -1];
>> v3=[2 3 5 -2];
>> v4=[3 4 -5 -1];

>> u1=v1/(norm(v1))

u1 =

   (0.5774         0    0.5774    0.5774)

>> up2=v2-v2*u1'*u1

up2 =
```

$u_2' =$

```
   (1.3333    3.0000   -0.6667   -0.6667)

>> u2=up2/(norm(up2))

u2 =

   (0.3904    0.8783   -0.1952   -0.1952)
```

```
>> up3=v3-v3*u1'*u1-v3*u2'*u2
```

up3 = u_3'

$$\begin{pmatrix} -0.7714 & 0.5143 & 3.8857 & -3.1143 \end{pmatrix}$$

```
>> u3=up3/(norm(up3))
```

u3 =

$$\begin{pmatrix} -0.1523 & 0.1015 & 0.7671 & -0.6148 \end{pmatrix}$$

```
>> up4=v4-v4*u1'*u1-v4*u2'*u2-v4*u3'*u3
```

up4 = u_4'

$$\begin{pmatrix} 1.2160 & -0.8107 & -0.3474 & -0.8686 \end{pmatrix}$$

```
>> u4=up4/(norm(up4))
```

u4 =

$$\begin{pmatrix} 0.7008 & -0.4672 & -0.2002 & -0.5006 \end{pmatrix}$$

(c) Define

$$Q = (u_1 \ u_2 \ u_3 \ u_4)$$

It follows that

$$BQ = Q\Lambda$$

where Λ is defined in page 1 and

Q is an orthogonal matrix ie

$$Q^{-1} = Q^T$$

Hence

$$B = Q \Lambda Q^T$$

Clearly B is symmetric because

$$B^T = Q \Lambda^T Q^T = B$$

Eigenvalue of B are all positive because

they are at $5, 7, 13, 17$. Hence B is

positive, definite.

```
>> Q=[u1' u2' u3' u4']

Q =
```

$$\begin{pmatrix} 0.5774 & 0.3904 & -0.1523 & 0.7008 \\ 0 & 0.8783 & 0.1015 & -0.4672 \\ 0.5774 & -0.1952 & 0.7671 & -0.2002 \\ 0.5774 & -0.1952 & -0.6148 & -0.5006 \end{pmatrix}$$

```
>> Q*Q'

ans =

    1.0000    0.0000   -0.0000    0.0000
    0.0000    1.0000    0.0000   -0.0000
   -0.0000    0.0000    1.0000    0.0000
    0.0000   -0.0000    0.0000    1.0000

>> lambda=[5 0 0 0;0 7 0 0;0 0 13 0;0 0 0 17]

lambda =
```

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 17 \end{pmatrix} = \Lambda$$

```
>> B=Q*lambda*Q'

B =
```

$$= Q \Lambda Q^T$$

$$\begin{pmatrix} 11.3834 & -3.3667 & -2.7708 & -3.6126 \\ -3.3667 & 9.2445 & 1.4027 & 1.9640 \\ -2.7708 & 1.4027 & 10.2651 & -2.4943 \\ -3.6126 & 1.9640 & -2.4943 & 11.1070 \end{pmatrix}$$

```
>> 2*B

ans =

   22.7668   -6.7334   -5.5415   -7.2253
   -6.7334   18.4890    2.8055    3.9280
   -5.5415    2.8055   20.5302   -4.9887
   -7.2253    3.9280   -4.9887   22.2140
```

(d) Define

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

We have

$$\underline{X}^T B \underline{X} = 100$$

$$11.38x_1^2 + 9.24x_2^2 + 10.26x_3^2 + 11.11x_4^2 - 6.73x_1x_2 - 5.54x_1x_3 - 7.22x_1x_4 + 2.80x_2x_3 + 3.93x_2x_4 - 4.99x_3x_4 = 100$$

$$\Rightarrow \underline{X}^T Q \Lambda Q^T \underline{X} = 100$$

If we now define

$$Y = Q^T \underline{X}$$

where

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}$$

we obtain

$$Y^T \Lambda Y = 100$$

e Substituting

$$\underline{X} = \begin{pmatrix} 3t \\ 4t \\ -5t \\ -t \end{pmatrix}$$

in the equation $\underline{X}^T B \underline{X} = 100$,

we obtain

```
>> syms t
>> X=[3*t 4*t -5*t -t]

X =

[  3*t,  4*t, -5*t,   -t]

>> X*B*X'

ans =

15758917003101376599317/35387034072063672320*t*conj(t)

>> 15758917003101376599317/35387034072063672320

ans =

  445.3303
```

$$445.33\, t^2 = 100$$

```
>> t=sqrt(100/445.3303)

t =

    0.4739
```

$$t = \pm .4739$$

```
>> eval(X)

ans =

  (1.4216    1.8955    -2.3693    -0.4739)
```

The two pts of intersection are $\pm(1.42 \quad 1.89 \quad -2.37 \quad -.47)$

(f) Let

$$\phi(\underline{X}) = \underline{X}^T B \underline{X}$$

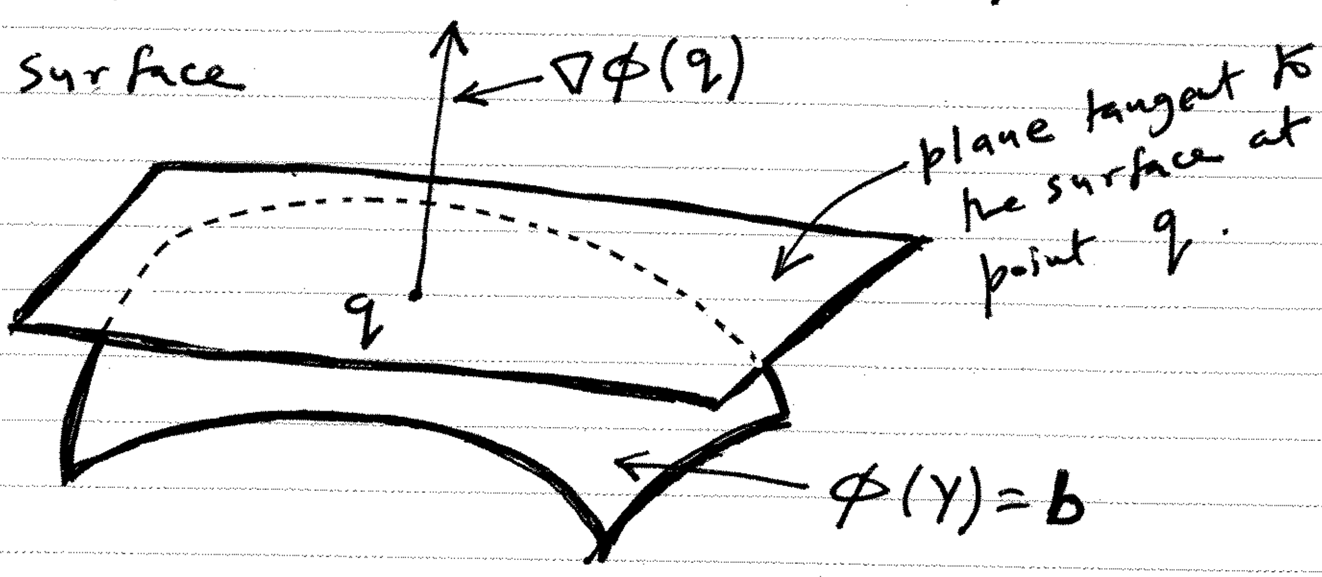
$$\nabla \phi(\underline{X}) = \left(\frac{\partial \phi}{\partial x_1} \quad \frac{\partial \phi}{\partial x_2} \quad \frac{\partial \phi}{\partial x_3} \quad \frac{\partial \phi}{\partial x_4} \right)$$

Let

$$\phi(Y) = Y^T \Lambda Y = 5Y_1^2 + 7Y_2^2 + 13Y_3^2 + 17Y_4^2$$

$$\nabla \phi(Y) = \left(10Y_1 \quad 14Y_2 \quad 26Y_3 \quad 34Y_4 \right)$$

$\nabla \phi$ is called the gradient vector and it can be shown that this vector is perpendicular to the tangent plane to the surface



Let
$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

be a point on the surface

$$\phi(Y) = 100.$$

$$\nabla\phi(q) = \begin{pmatrix} 10q_1 \\ 14q_2 \\ 26q_3 \\ 34q_4 \end{pmatrix}$$

is perpendicular to the tangent plane to the surface $\phi(Y) = 100$ at the point q.

It follows that the equation of the tangent plane must be

$$10q_1Y_1 + 14q_2Y_2 + 26q_3Y_3 + 34q_4Y_4 = c$$

where c is a constant to be determined so that the plane actually passes through q. It follows that

$$10 q_1^2 + 14 q_2^2 + 26 q_3^2 + 34 q_4^2 = c$$

obtained by substituting $Y = q$ in the plane equation.

Thus, substituting c back into the equation we have

$$10 q_1 Y_1 + 14 q_2 Y_2 + 26 q_3 Y_3 + 34 q_4 Y_4$$
$$= 10 q_1^2 + 14 q_2^2 + 26 q_3^2 + 34 q_4^2$$

$$\Rightarrow \boxed{10 q_1 (Y_1 - q_1) + 14 q_2 (Y_2 - q_2) + 26 q_3 (Y_3 - q_3) + 34 q_4 (Y_4 - q_4) = 0}$$

Equation of the tangent plane in the Y co-ordinate

In vector notation, the tangent plane is written as

$$2\, q^T \Lambda (y - q) = 0$$

$$\Rightarrow \boxed{q^T \Lambda (y - q) = 0}$$

where

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \; ; \; y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \; ; \; \Lambda = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 17 \end{pmatrix}$$

Changing back to the $\underline{X}$ co-ordinate, see page 4, recall that

$$\underline{X} = Q\, y .$$

Define $p = Q\, q$. We would like to calculate the tangent plane at p in the co-ordinate system $\underline{X}$.

Substituting

$$\underline{X} = Q y, \quad p = Q q$$

into

$$q^T \Lambda (y - q) = 0$$

Note that $q = Q^T p$ $\Rightarrow q^T = p^T Q$.

we obtain.

$$\underbrace{(p^T Q)}_{q^T} \Lambda \left(\underbrace{Q^T \underline{X}}_{y} - \underbrace{Q^T p}_{q} \right) = 0$$

$$\Rightarrow p^T \underbrace{(Q \Lambda Q^T)}_{B} (\underline{X} - p) = 0$$

$$\Rightarrow p^T B (\underline{X} - p) = 0$$

is the equation of the tangent plane to the surface $\underline{X}^T B \underline{X} = 100$ at the point p.

9 writing

$\phi(\underline{X}) =$

$$\underbrace{(8 \quad 9 \quad 7 \quad -12)}_{\theta^T} \underline{X}$$

$$= \theta^T \underline{X} = \theta^T Q y$$

Define $\theta_0^T = \theta^T Q$, we have a new function

$$\phi(y) = \theta_0^T y.$$

If $\underline{X}$ belongs to the ellipsoid $\underline{X}^T B \underline{X} = 100$,

$y = Q^T \underline{X}$ belongs to the ellipsoid $y^T \Lambda y = 100$

Problem:

Maximize or Minimize $\theta_0^T y$

subject to the constraint

$$y^T \Lambda y = 100 \Rightarrow y^T \Lambda y - 100 = 0$$

Define a plane in $\mathbb{R}^4$ with coordinates (y_1, y_2, y_3, y_4) as follows

$$\theta_0^T y = b \quad (*)$$

Where b is an arbitrary constant.

At the point q where the function $\phi(y) = \theta_0^T y$ is maximized or minimized, the plane $(*)$ is tangent to the ellipsoid $y^T \Lambda y = 100$. We do not know q but we know that

(1) q belongs to the ellipsoid $y^T \Lambda y = 100$

(2) The gradient vector $\begin{pmatrix} 10q_1 \\ 14q_2 \\ 26q_3 \\ 34q_4 \end{pmatrix}$ to the ellipsoid at q must point in the same direction as the vector θ_0.

Writing

$$\theta_0 = \begin{pmatrix} \theta_0^{(1)} \\ \theta_0^{(2)} \\ \theta_0^{(3)} \\ \theta_0^{(4)} \end{pmatrix}$$

it follows that

$$q_1 = \frac{\theta_0^{(1)}}{10}\lambda$$

$$q_2 = \frac{\theta_0^{(2)}}{14}\lambda \qquad \text{for some scalar } \lambda.$$

$$q_3 = \frac{\theta_0^{(3)}}{26}\lambda$$

$$q_4 = \frac{\theta_0^{(4)}}{34}\lambda$$

Since $5q_1^2 + 7q_2^2 + 13q_3^2 + 17q_4^2 = 100$

we have

$$\left[\left(\frac{\theta_0^{(1)}}{10}\right)^2 + \left(\frac{\theta_0^{(2)}}{14}\right)^2 + \left(\frac{\theta_0^{(3)}}{26}\right)^2 + \left(\frac{\theta_0^{(4)}}{34}\right)^2\right] \times$$

$$42\lambda^2 = 100$$

Thus λ can be calculated as

$$\lambda = \pm \frac{10}{\sqrt{42}} \frac{1}{\sqrt{\left(\frac{\theta_0^{(1)}}{10}\right)^2 + \left(\frac{\theta_0^{(2)}}{14}\right)^2 + \left(\frac{\theta_0^{(3)}}{26}\right)^2 + \left(\frac{\theta_0^{(4)}}{34}\right)^2}}$$

The two values of λ give maximum and minimum.

```
>> theta=[8 9 7 -12]

theta =

     8     9     7   -12
```

$\leftarrow \theta^T$

```
>> theta0=theta*Q

theta0 =
```

θ_0^T

```
   (1.7321   12.0036   12.4432    6.0067)

>> ll=(theta0(1)/10)^2+(theta0(2)/14)^2+(theta0(3)/26)^2+(theta0(4)/34)^2

ll =

    1.0254

>> llambda=10/(sqrt(42)*ll)
```

$\leftarrow \lambda$

```
llambda =

    1.5048
```

$= \lambda$

```
>> qq1=theta0(1)*llambda/10

qq1 =

    0.2606
```

$$q_1 = \frac{\theta_0^{(1)}}{10}\lambda$$

```
>> qq2=theta0(2)*llambda/14

qq2 =

    1.2902
```

$$q_2 = \frac{\theta_0^{(2)}}{14}\lambda$$

```
>> qq3=theta0(3)*llambda/26

qq3 =

    0.7202
```

$$q_3 = \frac{\theta_0^{(3)}\lambda}{26}$$

```
>> qq4=theta0(4)*llambda/34

qq4 =

    0.2659
```

$$q_4 = \frac{\theta_0^{(4)}\lambda}{34}$$

The function $\phi(Y) = \theta_0^T Y$ attains a maximum at

$$Y = (\cdot 2606 \quad 1{\cdot}2902 \quad \cdot 7202 \quad \cdot 2659)^T$$

and minimum at

$$Y = (-\cdot 2606 \quad -1{\cdot}2902 \quad -\cdot 7202 \quad -\cdot 2659)^T$$

Value of the function is $\pm 26{\cdot}5$

Likewise, the function $\phi(x) = \theta^T \underline{X}$

attains a maximum at

$$\underline{X} = Q\,Y = (\cdot 7308 \quad 1\cdot 0821 \quad \cdot 3979 \quad -\cdot 6772)$$

a minimum at

$$\underline{X} = -Q\,Y = (-\cdot 7308 \quad -1\cdot 0821 \quad -\cdot 3979 \quad \cdot 6772)$$

Value of the function is $\pm 26\cdot 5$

2 Ans:

a In order to get $\sin 2t$, we need an o.d.e. of the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and in order to get $\sin 4t$, we need an o.d.e. of the form

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Solving the above equations we get

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\begin{pmatrix} x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} \cos 4t & \sin 4t \\ -\sin 4t & \cos 4t \end{pmatrix} \begin{pmatrix} x_3(0) \\ x_4(0) \end{pmatrix}$$

Choose A as

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & -4 & 0 \end{pmatrix}$$

$x_1(0) = 0$, $x_2(0) = 1$, $x_3(0) = 0$, $x_4(0) = 1$.

$$\mathbf{x}(t) = \begin{pmatrix} \sin 2t \\ \cos 2t \\ \sin 4t \\ \cos 4t \end{pmatrix}$$

— x —

(b) In order to get $e^{-3t}\sin 7t$ we need an ode of the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 7 \\ -7 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \begin{matrix} x_1(0) = 0 \\ x_2(0) = 1. \end{matrix}$$

with eigenvalues at $-3 \pm 7i$.

In order to get $te^{-3t}\sin 7t$, the eigenvalues have to repeat twice, we would have an A matrix of the form

$$A = \begin{pmatrix} -3 & 7 & 1 & 0 \\ -7 & -3 & 0 & 1 \\ 0 & 0 & -3 & 7 \\ 0 & 0 & -7 & -3 \end{pmatrix}$$

The ode

$$\dot{\underline{x}} = A\,\underline{x}$$

has sol^{n}

$$\underline{X}(t) = e^{At}\,\underline{X}(0)$$

where

$$e^{At} = \begin{pmatrix} e^{-3t}\cos 7t & e^{-3t}\sin 7t & te^{-3t}\cos 7t & te^{-3t}\sin 7t \\ -e^{-3t}\sin 7t & e^{-3t}\cos 7t & -te^{-3t}\sin 7t & te^{-3t}\cos 7t \\ 0 & 0 & e^{-3t}\cos 7t & e^{-3t}\sin 7t \\ 0 & 0 & -e^{-3t}\sin 7t & e^{-3t}\cos 7t \end{pmatrix}$$

If $\underline{X}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, we obtain

$$\underline{X}(t) = \begin{pmatrix} te^{-3t}\sin 7t \\ te^{-3t}\cos 7t \\ e^{-3t}\sin 7t \\ e^{-3t}\cos 7t \end{pmatrix}.$$

3 Ans:

$$\ddot{x} + b\dot{x} + x = \sin 4t$$

The solution $x(t)$ is the sum of

$$\underbrace{x_h(t)}_{\text{homogeneous}} + \underbrace{x_p(t)}_{\text{particular}}$$

$\boxed{\text{Particular Sol}^{\underline{n}} \text{ (Depends only on the forcing term } \sin 4t)}$

$$x_p(t) = c_1 \sin 4t + c_2 \cos 4t$$

To find c_1 & c_2 we substitute $x_p(t)$ into the equation.

$$\dot{x}_p(t) = 4c_1 \cos 4t - 4c_2 \sin 4t$$

$$\ddot{x}_p(t) = -16c_1 \sin 4t - 16c_2 \cos 4t$$

It follows that

$$-16c_1 \sin 4t \quad -16c_2 \cos 4t$$
$$-4bc_2 \sin 4t \quad +4bc_1 \cos 4t \quad = \sin 4t$$
$$+c_1 \sin 4t \quad + c_2 \cos 4t$$

$$\Rightarrow -16c_1 - 4bc_2 + c_1 = 1$$
$$-16c_2 + 4bc_1 + c_2 = 0$$

$$\Rightarrow -15c_1 - 4bc_2 = 1$$
$$-15c_2 + 4bc_1 = 0$$

$$\begin{pmatrix} -15 & -4b \\ 4b & -15 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$c_1 = \frac{-15}{16b^2+225}, \quad c_2 = \frac{-4b}{16b^2+225}$$

$$x_p(t) = \frac{-15\sin 4t - 4b\cos 4t}{16b^2+225}$$

$\boxed{\text{Homogeneous Sol}^{\underline{n}} \text{ (Depends only on the characteristic polynomial}}$

$$\lambda^2 + b\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4}}{2}$$

(a) $b = 1$ $\quad \lambda = \dfrac{-1 \pm \sqrt{-3}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$

(b) $b = 3$ $\quad \lambda = \dfrac{-3 \pm \sqrt{5}}{2} = \dfrac{-3 \pm \sqrt{5}}{2}$

$$\lambda_1 = \frac{-3 + \sqrt{5}}{2} \qquad \lambda_2 = \frac{-3 - \sqrt{5}}{2}$$

(c) $b = 2$ $\quad \lambda = \dfrac{-2 \pm \sqrt{0}}{2} = -1$

(a) $b=1$

$$x_h(t) = d_1 e^{-t/2} \cos\frac{\sqrt{3}}{2}t + d_2 e^{-t/2} \sin\frac{\sqrt{3}}{2}t$$

$$x(t) = x_p(t) + x_h(t)$$

$$= \frac{-15\sin 4t - 4\cos 4t}{241}$$

$$+ \quad e^{-t/2}\left(d_1 \cos\frac{\sqrt{3}}{2}t + d_2 \sin\frac{\sqrt{3}}{2}t\right)$$

To compute d_1 & d_2 we observe that

$$x(0) = 5, \quad \dot{x}(0) = 15$$

$$x(0) = -\frac{4}{241} + d_1 = 5 \Rightarrow d_1 = 5 + \frac{4}{241}$$

$$\boxed{d_1 = \frac{1209}{241}} \qquad = \frac{1205+4}{241}$$

$$\dot{x}(0) = -\frac{60}{241} + e^{-t/2}\left(d_2 \frac{\sqrt{3}}{2}\right)\Bigg|_{t=0}$$

$$- \frac{1}{2} e^{-t/2} d_1 \Bigg|_{t=0}$$

$$= -\frac{60}{241} + \frac{\sqrt{3}}{2} d_2 - \frac{1}{2} d_1 = 15$$

$$\frac{\sqrt{3}}{2} d_2 = \frac{60}{241} + \frac{1}{2}\,\frac{1209}{241} + 15$$

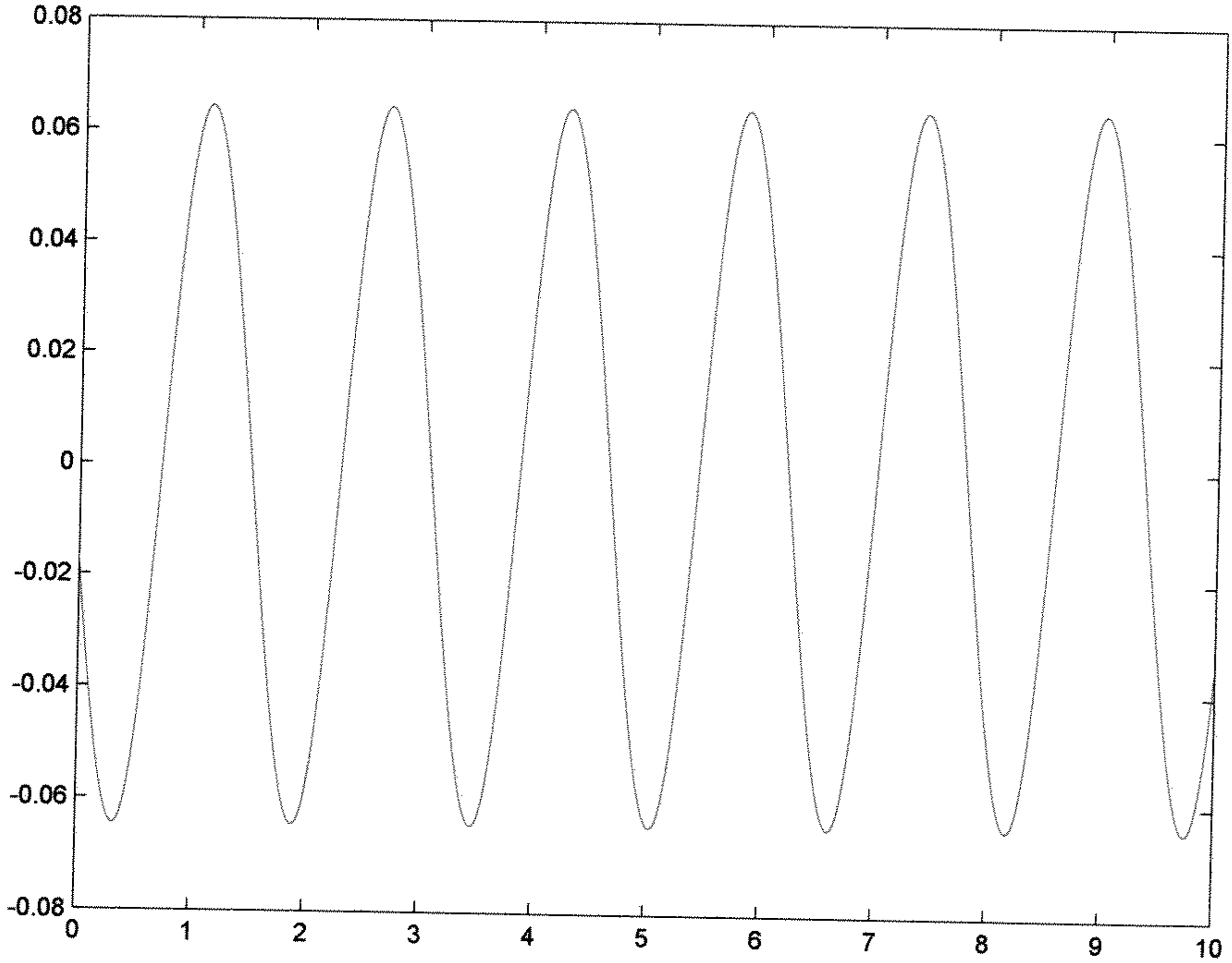
$$= \frac{120 + 1209 + 7230}{2 \cdot 241}$$

$$d_2 = \frac{8559}{241\sqrt{3}}$$

$$X(t) = -\frac{15}{241}\sin 4t - \frac{4}{241}\cos 4t$$

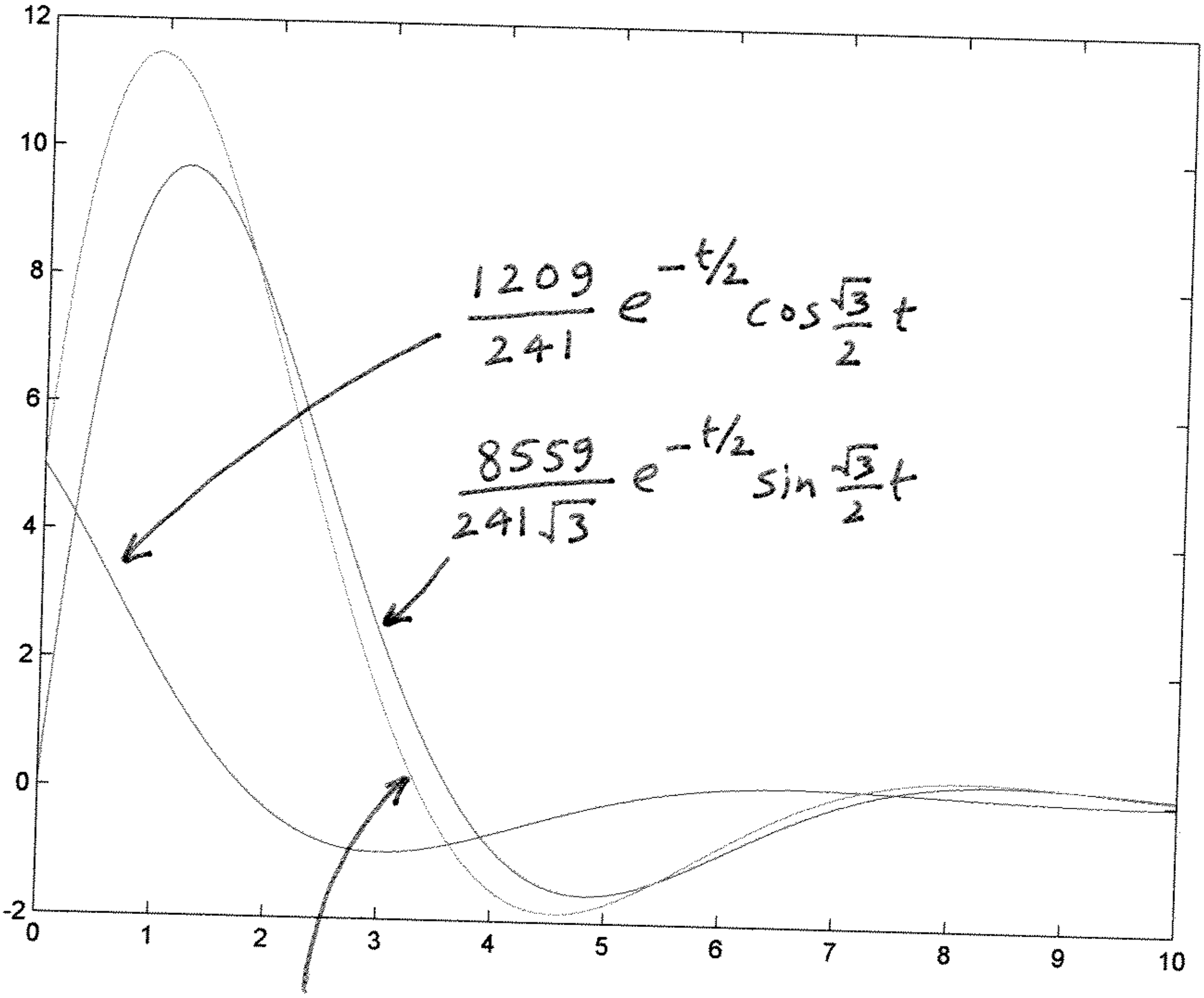
$$+ \frac{1}{241} e^{-t/2}\left(1209\cos\frac{\sqrt{3}}{2}t + \frac{8559}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$

—— X ——

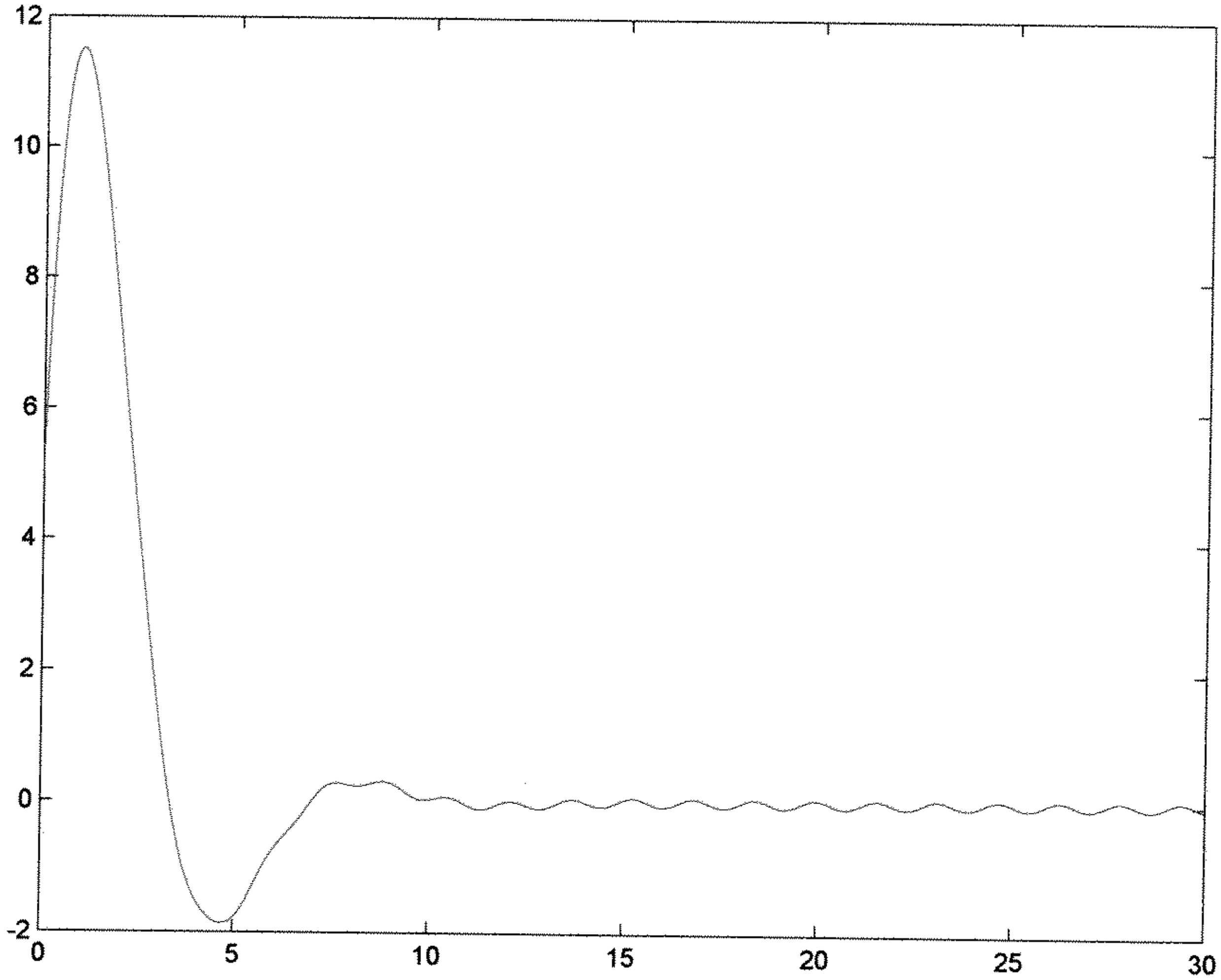


Forced Response

$$x(t) = -\frac{15}{241}\sin 4t - \frac{4}{241}\cos 4t$$



Natural Response is the sum of the two functions.



The plot of $x(t)$. Note that it takes about 7 units of time for the natural response to die out.

(b) $b = 3$

$$x_h(t) = d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t}$$

$$x(t) = x_p(t) + x_h(t)$$

$$= \frac{-15\sin 4t - 12\cos 4t}{369} + d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t}$$

$$x(0) = -\frac{12}{369} + d_1 + d_2 = 5$$

$$\Rightarrow d_1 + d_2 = 5 + \frac{12}{369} = \frac{1857}{369}$$

$$\dot{x}(0) = -\frac{60}{369} + \lambda_1 d_1 + \lambda_2 d_2 = 15$$

$$\lambda_1 d_1 + \lambda_2 d_2 = 15 + \frac{60}{369} = \frac{5535+60}{369}$$

$$= \frac{5595}{369}$$

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1857 \\ 5595 \end{pmatrix} \Big/ 369$$

$$d_1 = \frac{1857\lambda_2 - 5595}{\lambda_2 - \lambda_1} \cdot \frac{1}{369}$$

$$d_2 = \frac{5595 - 1857\lambda_1}{\lambda_2 - \lambda_1} \frac{1}{369}$$

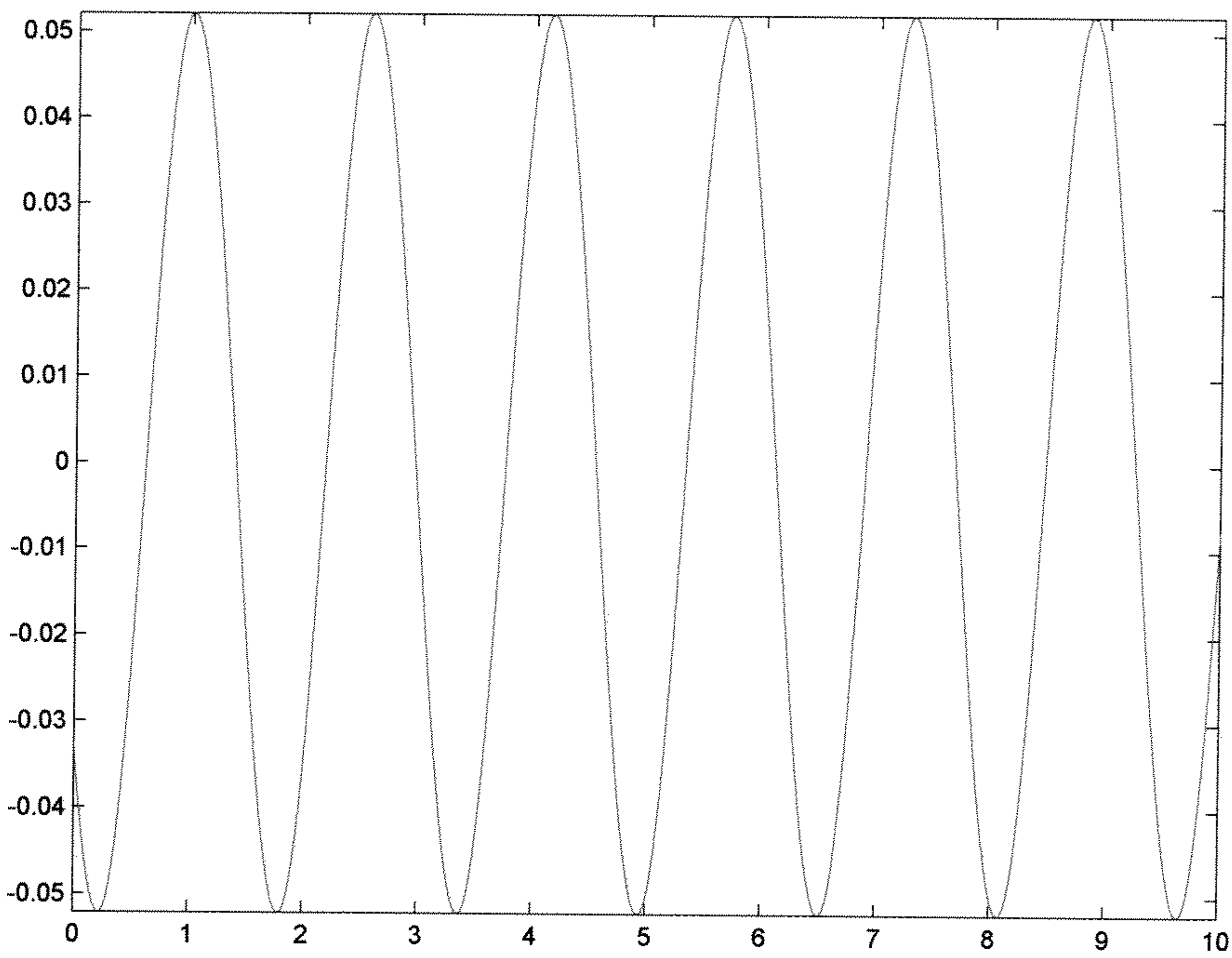
$$\lambda_2 - \lambda_1 = -\frac{\sqrt{5}}{2} - \frac{\sqrt{5}}{2} = -\sqrt{5}$$

$$d_1 = \frac{1}{369\sqrt{5}} \left(5595 - 1857\lambda_2\right)$$

$$d_2 = \frac{1}{369\sqrt{5}} \left(1857\lambda_1 - 5595\right)$$

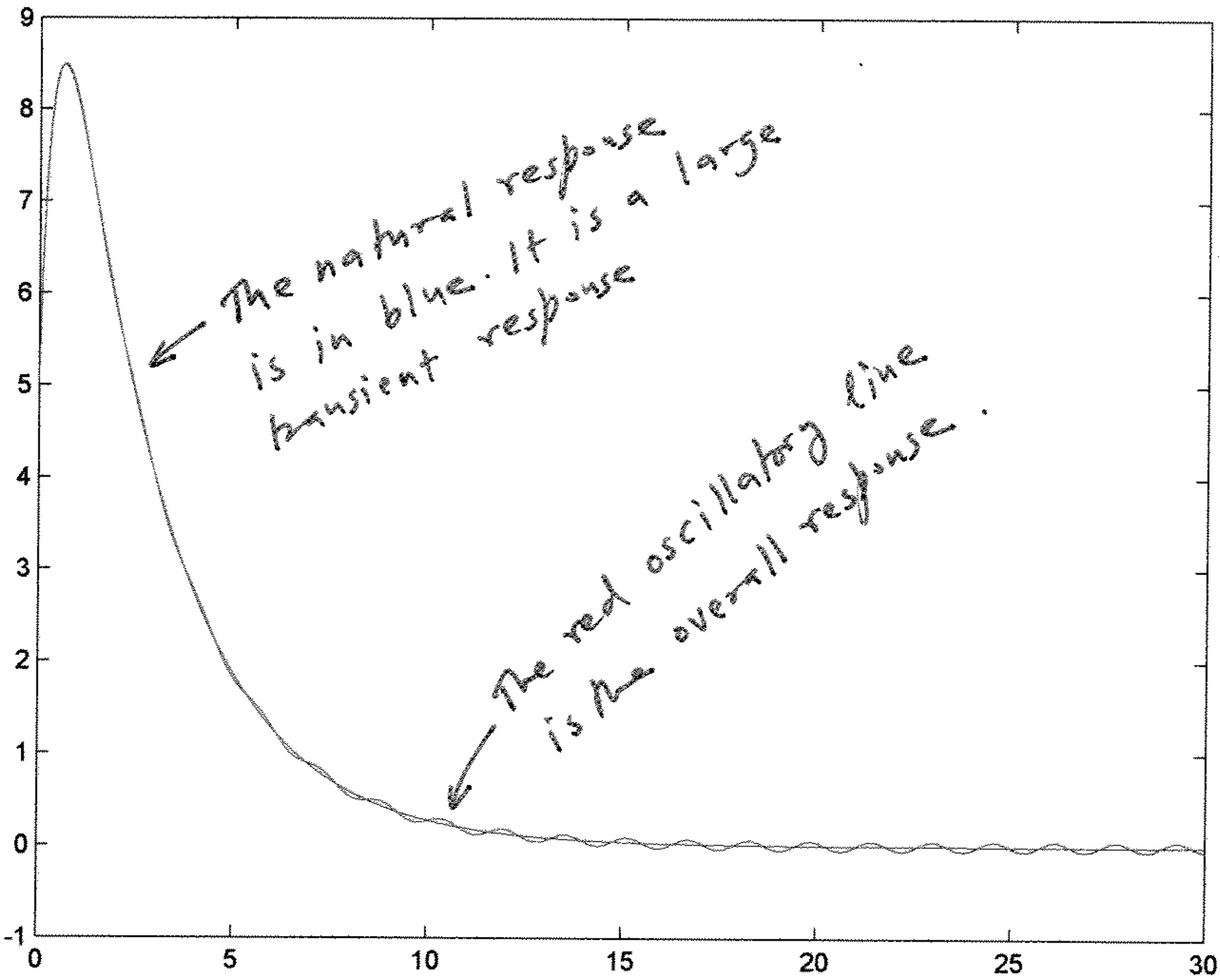
$$x(t) = \frac{-15}{369}\sin 4t - \frac{12}{369}\cos 4t$$

$$+ \frac{1}{369\sqrt{5}} \underbrace{\begin{pmatrix} 5595 e^{\lambda_1 t} - 1857\lambda_2 e^{\lambda_1 t} \\ +1857\lambda_1 e^{\lambda_2 t} - 5595 e^{\lambda_2 t} \end{pmatrix}}_{\substack{\parallel \\ 5595\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) \\ +1857\left(\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}\right)}}$$



The forced response

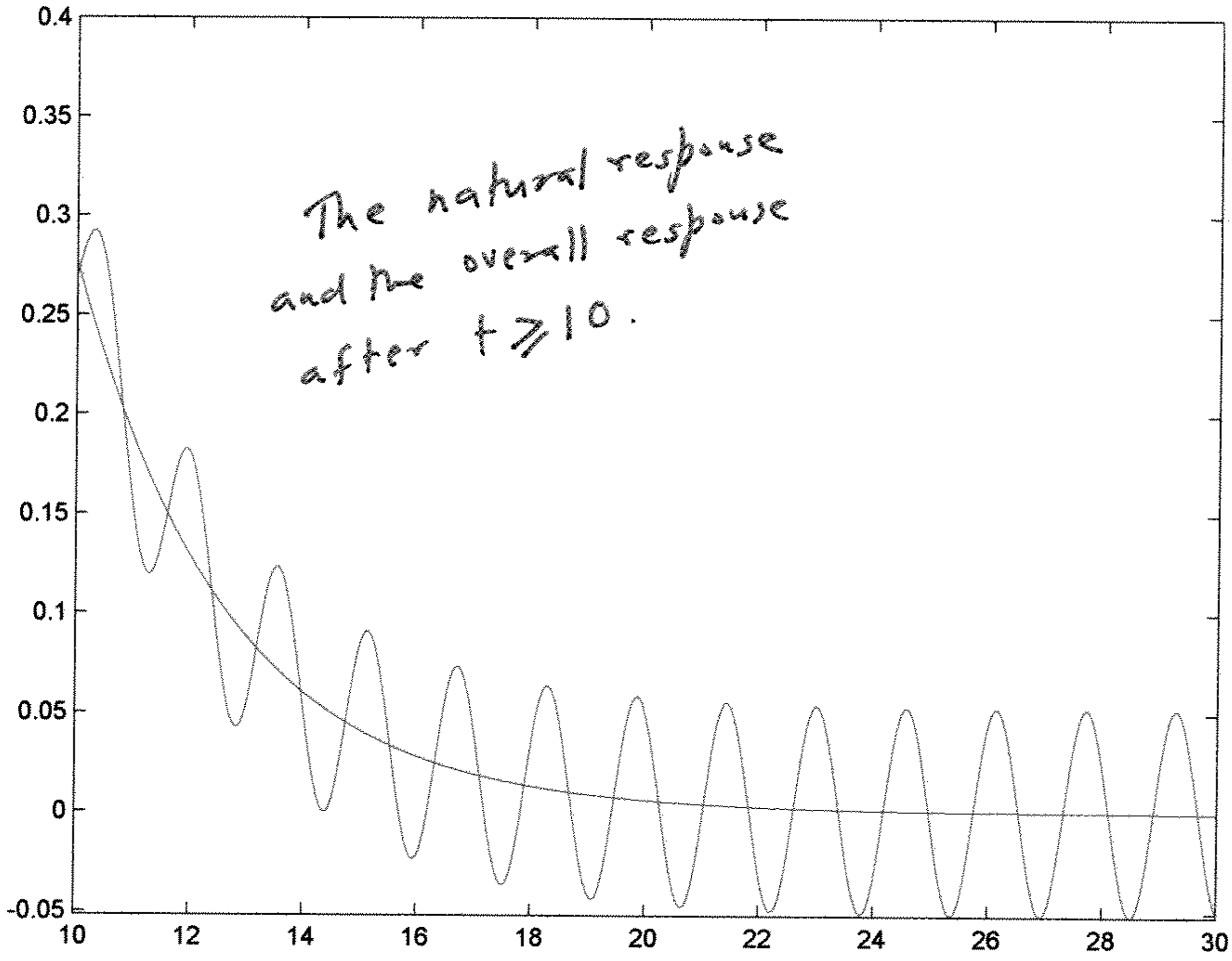
$$x(t) = -\frac{15}{369}\sin 4t - \frac{12}{369}\cos 4t$$



Natural response

$$\frac{5595}{369\sqrt{5}}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)$$

$$+\frac{1857}{369\sqrt{5}}\left(\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}\right).$$


The natural response
and the overall response
after t ≥ 10.
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
10 12 14 16 18 20 22 24 26 28 30

(c) $b=2$

$$x_h(t) = d_1 e^{-t} + d_2 t e^{-t}$$

$$x(t) = x_p(t) + x_h(t)$$

$$= \frac{-15\sin 4t - 8\cos 4t}{289}$$

$$+ (d_1 + d_2 t) e^{-t}$$

$$x(0) = -\frac{8}{289} + d_1 = 5$$

$$d_1 = 5 + \frac{8}{289} = \frac{1453}{289}$$

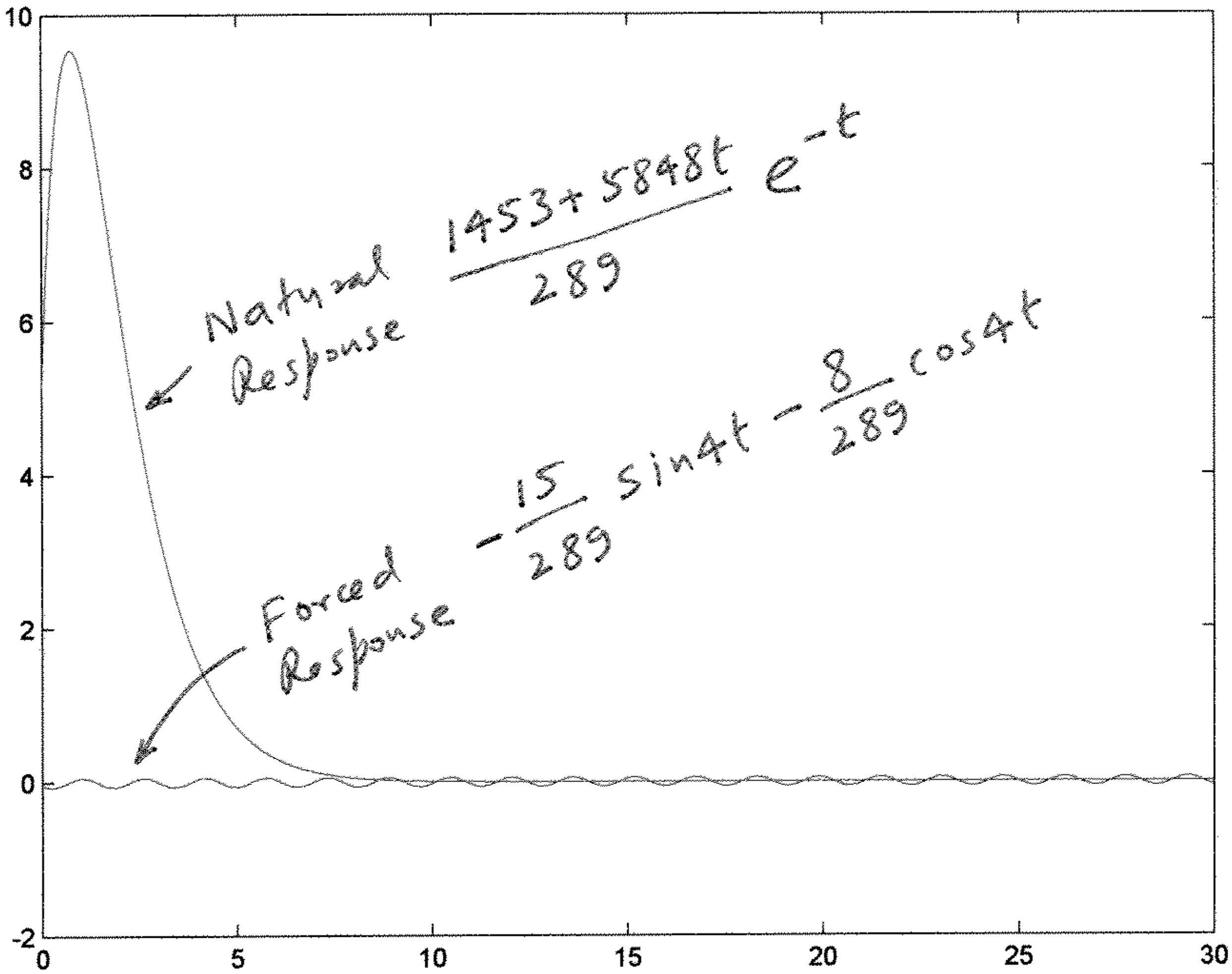
$$\dot{x}(0) = -\frac{60}{289} + (d_1 + d_2 t)(-e^{-t}) + d_2 e^{-t} \Big|_{t=0}$$

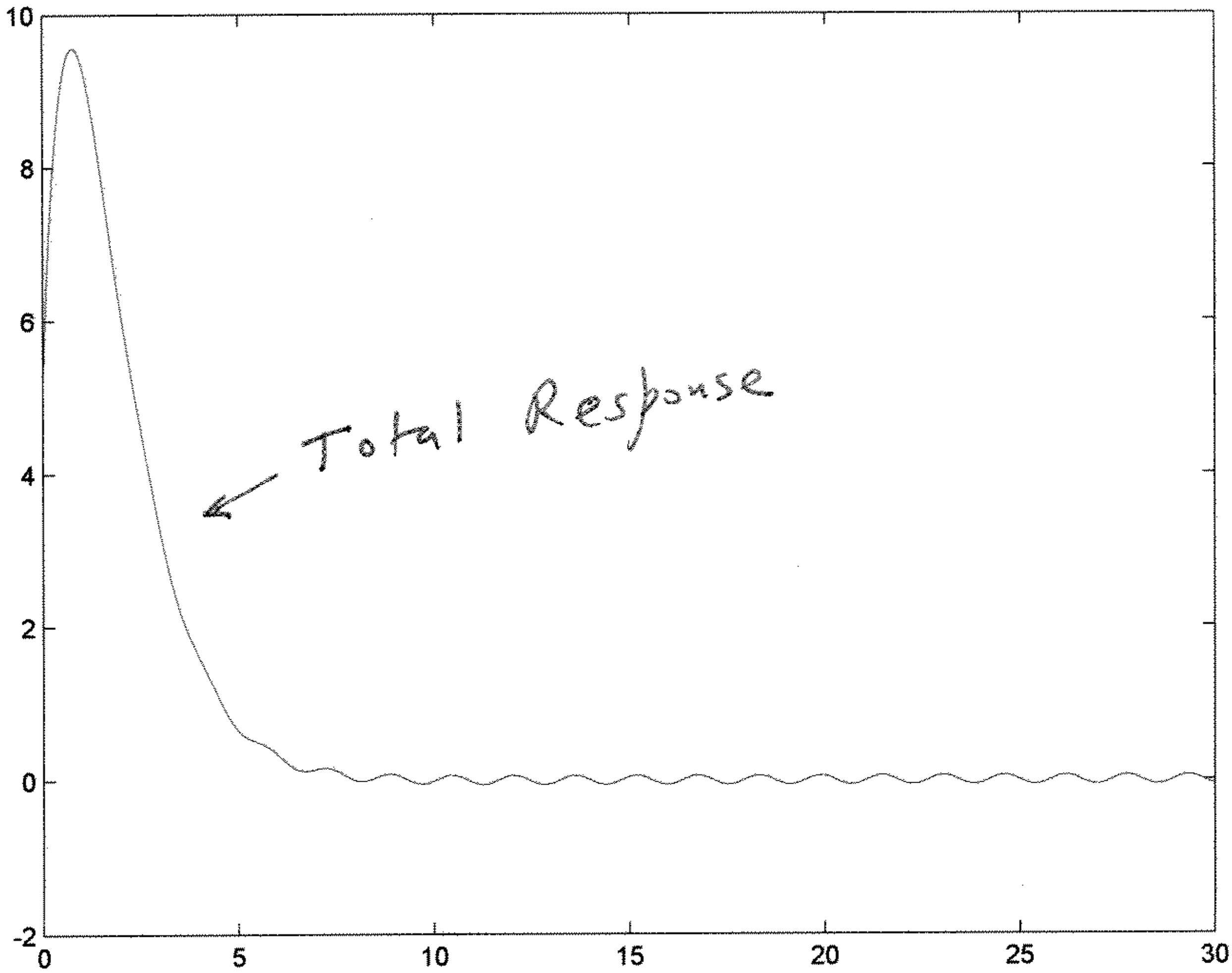
$$= -\frac{60}{289} + d_2 - d_1 = 15$$

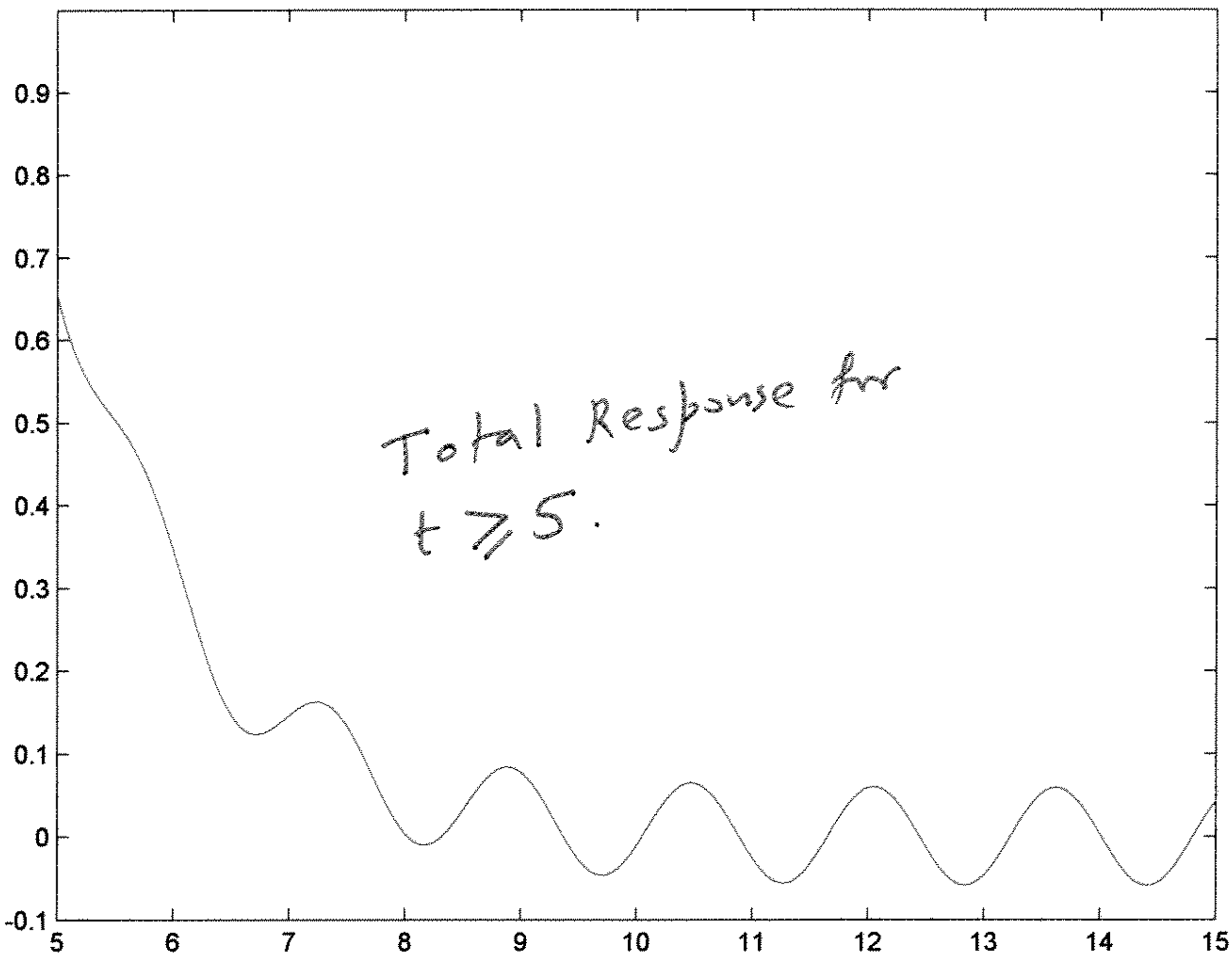
$$d_2 = \frac{60}{289} + \frac{1453}{289} + 15 = \frac{5848}{289}$$

38

$$x(t) = \frac{1}{289}\left(-15\sin 4t - 8\cos 4t + (1453 + 5848t)e^{-t}\right)$$


10
8
6
4
2
0
-2
0
5
10
15
20
25
30
Natural Response
$\frac{1453+5848t}{289}e^{-t}$
Forced Response
$-\frac{15}{289}\sin 4t - \frac{8}{289}\cos 4t$


Total Response
10
8
6
4
2
0
-2
0
5
10
15
20
25
30


Total Response for
t ≥ 5.
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
5
6
7
8
9
10
11
12
13
14
15

4

a

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\boxed{\text{Type: Unstable Node}}$

$$x_1(t) = e^{t} x_1(0)$$
$$x_2(t) = e^{2t} x_2(0)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1(0) + e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2(0)$$

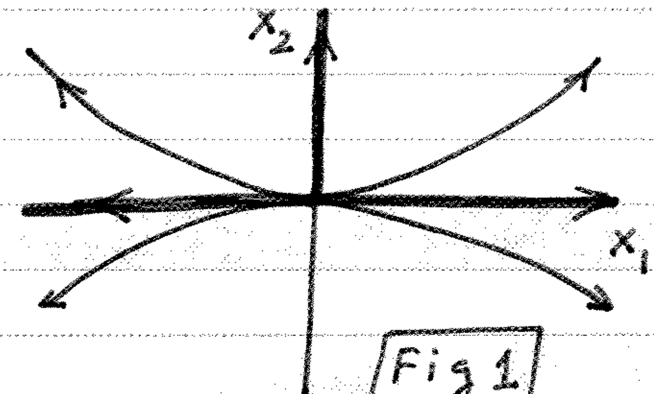


Fig 1

Type: Saddle point

(b) $$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}2 & 1\\ 5 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\text{Char pol} = (\lambda-2)(\lambda+2) - 5$$
$$= \lambda^2 - 9$$

$$\lambda = \pm 3$$

Calculate Eigenvector

$$\begin{pmatrix}2 & 1\\ 5 & -2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}3v_1\\ 3v_2\end{pmatrix}$$

$$\Rightarrow 2v_1 + v_2 = 3v_1$$
$$\Rightarrow \boxed{v_1 = v_2}$$

$$\lambda = 3 \quad v = \begin{pmatrix}1\\ 1\end{pmatrix}$$

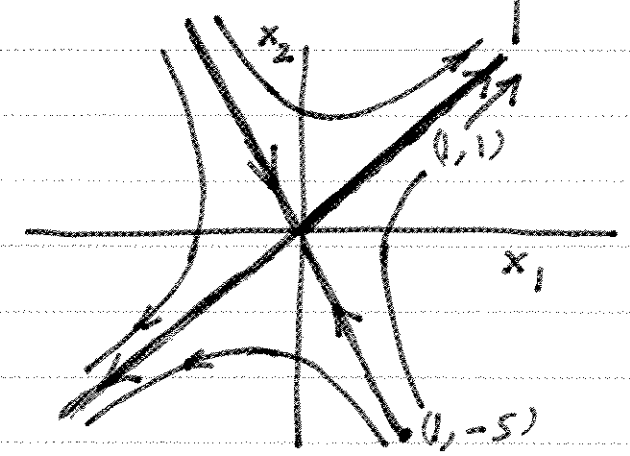
Eigenvector

$$\begin{pmatrix}2 & 1\\ 5 & -2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}-3v_1\\ -3v_2\end{pmatrix}$$

$$\Rightarrow 2v_1 + v_2 = -3v_1$$
$$\Rightarrow v_2 = -5v_1$$

$$\lambda = -3 \quad v = \begin{pmatrix}1\\ -5\end{pmatrix}$$

Eigenvector.



$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}(t) = e^{3t}\begin{pmatrix}1\\ 1\end{pmatrix}a + e^{-3t}\begin{pmatrix}1\\ -5\end{pmatrix}b.$$

(c) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Type:
Saddle Point

Char polynomial =

$$(\lambda-1)^2 - 4 = 0$$

$$\Rightarrow (\lambda-1)^2 = 4$$

$$\Rightarrow \lambda - 1 = \pm 2$$

$$\Rightarrow \lambda = 1 \pm 2 = 3, -1$$

Eigenvector calculation

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 \\ 3v_2 \end{pmatrix}$$

$\Rightarrow v_1 + 2v_2 = 3v_1$

$\Rightarrow v_1 = v_2$

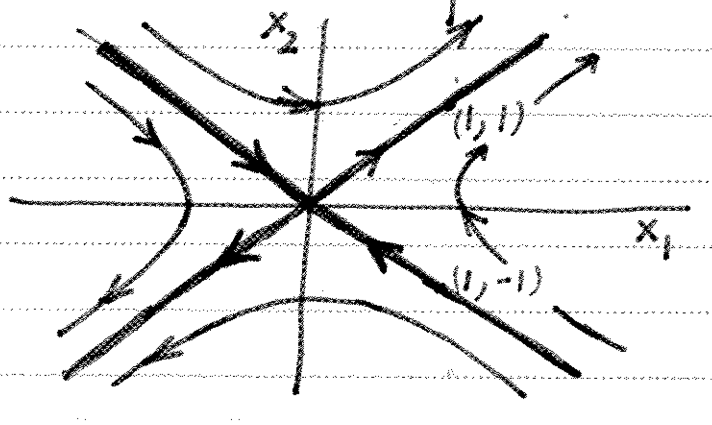
$\lambda = 3 \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$$

$\Rightarrow v_1 + 2v_2 = -v_1$

$\Rightarrow v_1 = -v_2$

$\lambda = -1 \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} a + e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} b$$

$\boxed{\text{Type: Stable Node}}$

(d) $$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-6 & -1\\ -9 & -6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Char polynomial =

$$(\lambda+6)^2 - 9 = 0$$

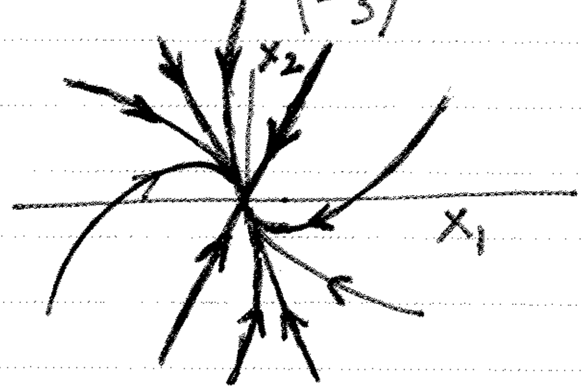
$$\Rightarrow (\lambda+6)^2 = 3^2$$

$$\Rightarrow (\lambda+6) = \pm 3$$

$$\Rightarrow \lambda = -6 \pm 3 = -9, -3$$

Eigenvector calculation	Eigenvector calculation.
$\boxed{\lambda=-9}$ $\begin{pmatrix}-6 & -1\\ -9 & -6\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}-9v_1\\ -9v_2\end{pmatrix}$	$\boxed{\lambda=-3}$ $\begin{pmatrix}-6 & -1\\ -9 & -6\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}-3v_1\\ -3v_2\end{pmatrix}$
$\Rightarrow -6v_1 - v_2 = -9v_1$	$\Rightarrow -6v_1 - v_2 = -3v_1$
$\Rightarrow v_2 = 3v_1$	$\Rightarrow v_2 = -3v_1$
$\lambda = -9$ $\begin{pmatrix}1\\ 3\end{pmatrix}$ ← Eigen vector.	$\lambda = -3$ $\begin{pmatrix}1\\ -3\end{pmatrix}$ ← Eigenvector.

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}(t) = a e^{-9t}\begin{pmatrix}1\\ 3\end{pmatrix} + b e^{-3t}\begin{pmatrix}1\\ -3\end{pmatrix}$$



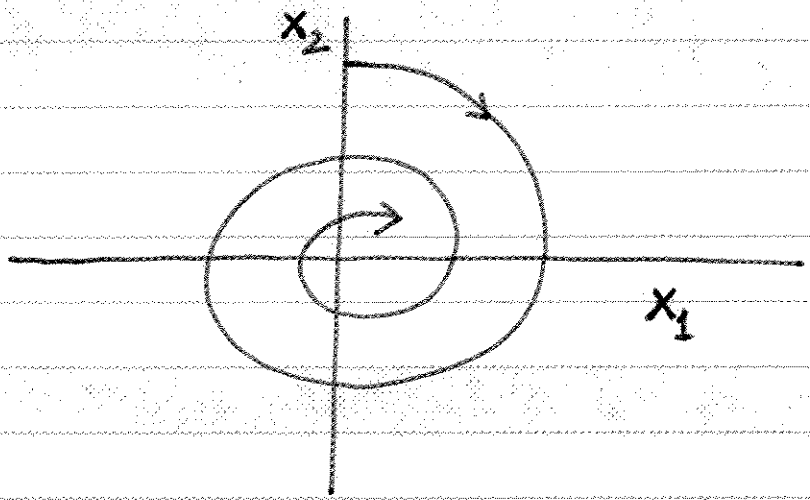
(e) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\boxed{\text{Type: Stable Focus}}$

Char poly $= (\lambda+2)^2 + 4 = 0$

$\Rightarrow (\lambda+2)^2 = -4 = (2i)^2$

$\Rightarrow \lambda = -2 \pm 2i$



$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-2t} \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} x_1(0)\cos 2t + x_2(0)\sin 2t \\ x_2(0)\cos 2t - x_1(0)\sin 2t \end{pmatrix}$$

$$\Rightarrow \sqrt{x_1^2(t) + x_2^2(t)} = e^{-2t}\sqrt{x_1^2(0) + x_2^2(0)}$$

Note: The phase portrait rotates clockwise because $\dot{x}_1(0) > 0$, $\dot{x}_2(0) < 0$ if $x_1(0) = 0$, $x_2(0) = 1$

(f) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\boxed{\text{Type: Center located at the origin}}$

Char poly $= (\lambda - 1)(\lambda + 1) + 10 = 0$

$\Rightarrow \lambda^2 - 1 + 10 = 0$

$\Rightarrow \lambda^2 = -9$

$\Rightarrow \lambda = \pm 3i$

$\exists$ a 2×2 matrix P :

$\underline{x} = PZ$

$\dot{\underline{x}} = P\dot{Z} = A\underline{x} = APZ$

$\dot{Z} = P^{-1}APZ$

$P^{-1}AP = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

$p_1 - 2p_3 = -3p_2 \qquad 5p_1 - p_3 = -3p_4$

$p_2 - 2p_4 = 3p_1 \qquad 5p_2 - p_4 = 3p_3$

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 3 & -1 & 0 & 2 \\ 5 & 0 & -1 & 3 \\ 0 & 5 & -3 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 0 & -3 & 9 \\ 0 & 5 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$15 p_1 = 3 p_3 - 9 p_4$$

$$5 p_2 = 3 p_3 + p_4$$

Choose $p_4 = 0 \quad p_3 = 5 \Rightarrow p_1 = 1, \; p_2 = 3$

$$P = \begin{pmatrix} 1 & 3 \\ 5 & 0 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 0 & 1/5 \\ 1/3 & -1/15 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/5 \\ 1/3 & -1/15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2/5 \\ x_1/3 - x_2/15 \end{pmatrix}$$

$$\boxed{\begin{aligned} 15 z_1 &= 3x_2 \\ 15 z_2 &= 5x_1 - x_2 \end{aligned}}$$

In the (z_1, z_2) co-ordinate the eqn is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}(t) = \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

$$z_1^2(t) + z_2^2(t) = \text{const.}$$

$$\Rightarrow (15 z_1)^2 + (15 z_2)^2 = \text{const.}$$

$$\Rightarrow (3x_2)^2 + (5x_1 - x_2)^2 = \text{const.}$$

$$\Rightarrow 9x_2^2 + 25x_1^2 + x_2^2 - 10x_1x_2 = \text{const.}$$

$$\Rightarrow 25x_1^2 + 10x_2^2 - 10x_1x_2 = \text{const.}$$

$$\Rightarrow (x_1 \;\; x_2) \begin{pmatrix} 25 & -5 \\ -5 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \text{const.} \quad (*)$$

Char. poly. of $\begin{pmatrix} 25 & -5 \\ -5 & 10 \end{pmatrix}$ is

$(\lambda-25)(\lambda-10)-25$

$= \lambda^2 - 35\lambda + 225$

orthonormal eigenvectors

$A = \begin{pmatrix} 25 & -5 \\ -5 & 10 \end{pmatrix} \quad P = \begin{pmatrix} .2898 & -.9571 \\ .9571 & .2898 \end{pmatrix}$

$P^T A P = \begin{pmatrix} 8.4861 & 0 \\ 0 & 26.5139 \end{pmatrix} = \Lambda$

P is an orthogonal matrix.

$A = P \Lambda P^T$

If $(y_1 \ \ y_2) = (x_1 \ \ x_2) P$, we have

$y_1 = .2898\, x_1 + .9571\, x_2$

$y_2 = -.9571\, x_1 + .2898\, x_2$

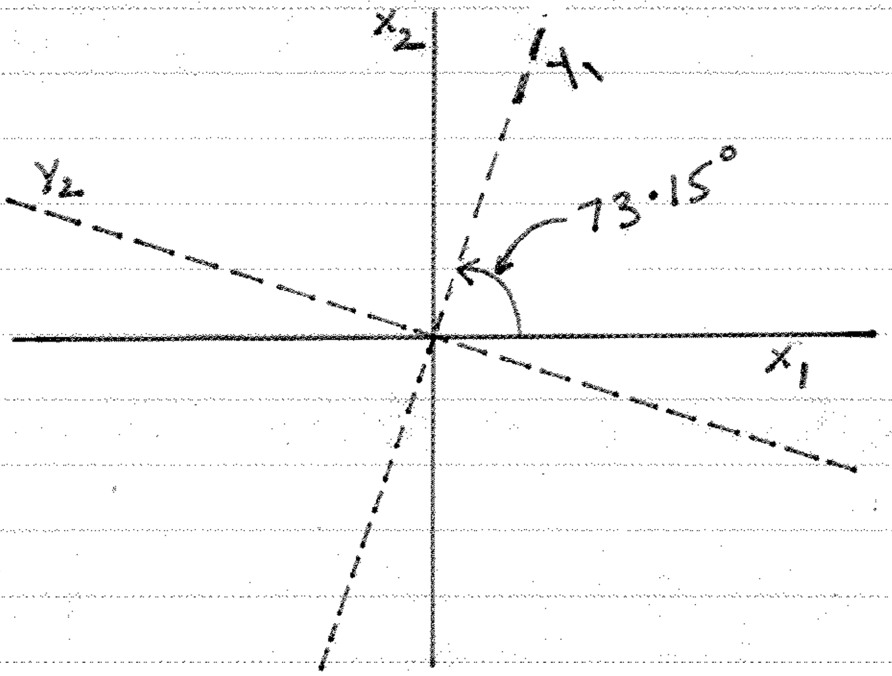
The ellipse (*) in the x_1, x_2 co-ordinate takes the form

$8.4861\, y_1^2 + 26.5139\, y_2^2 = \text{const}.$

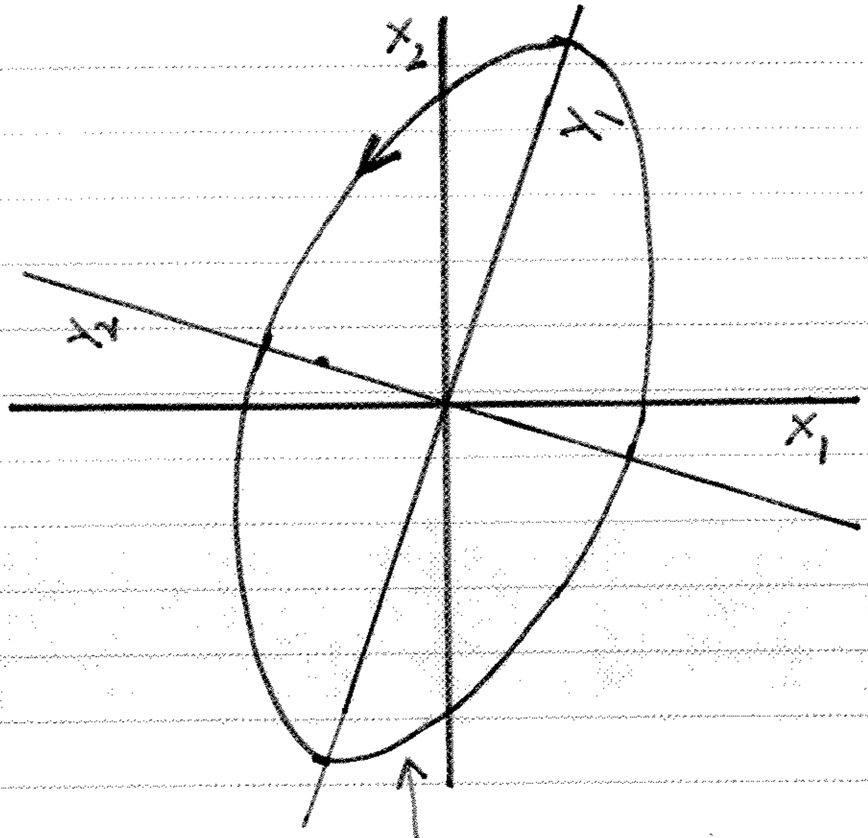
(51)

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \underset{\cos\theta}{\cdot 2898} & \underset{\sin\theta}{\cdot 9571} \\ -\cdot 9571 & \cdot 2898 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\theta = 73{\cdot}15^\circ.$$



The $(Y_1\ Y_2)$ co-ordinates are obtained by rotating $(x_1\ x_2)$ co-ordinates by 73·15°.



$$8.4861 y_1^2 + 26.5139 y_2^2 = 100$$

Phase portraits are elliptical and anticlockwise.

Note that when $x_1(0)=0, x_2(0)=1,$

$\dot{x}_1(0)<0, \ \dot{x}_2(0)<\underline{1}$.

This suggests anticlockwise orientation.

Type: Center at the origin

(g)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -9 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Assume $x_1(0) = 0 \quad x_2(0) = 3$

Char poly $= \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

$x_1(t) = a\cos 3t + b\sin 3t$

$0 = x_1(0) = a \Rightarrow x_1(t) = b\sin 3t$

$\dot{x}_2 = -9x_1 = -9b\sin 3t$

$x_2(t) = 3b\cos 3t + c$

$\because \dot{x}_1 = x_2$ it follows that $c = 0$.

Hence

$x_1(t) = b\sin 3t$

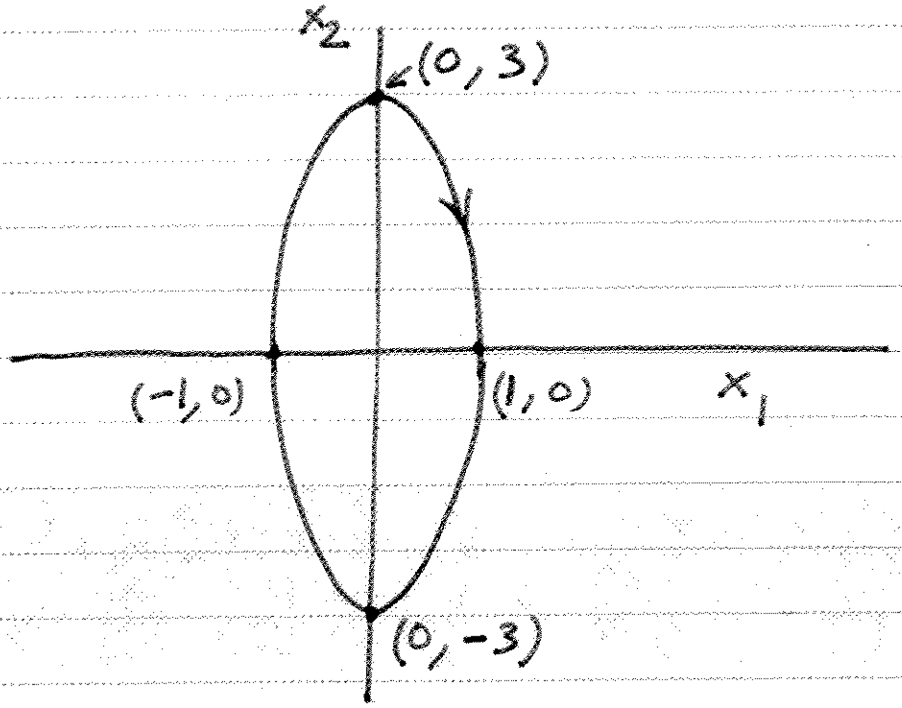
$x_2(t) = 3b\cos 3t$

$3 = x_2(0) = 3b \Rightarrow b = 1$

$$\boxed{\begin{array}{l} x_1(t) = \sin 3t \\ x_2(t) = 3\cos 3t \end{array}} \Rightarrow x_1^2 + \left(\frac{x_2}{3}\right)^2 = 1$$

$$x_1^2 + \frac{x_2^2}{9} = 1 \Rightarrow \boxed{9x_1^2 + x_2^2 = 9}$$

Ellipse



$\dot{x}_1(0) > 0$ since $\dot{x}_1 = x_2$ & $\dot{x}_1(0) = x_2(0) = 3$

The phase portrait is elliptical and rotates clockwise

$\boxed{\text{Type: Saddle Point}}$

(h) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Char poly $= (\lambda+1)(\lambda+2) - 12$

$= \lambda^2 + 3\lambda - 10$

$= (\lambda+5)(\lambda-2)$

$\lambda_1 = -5, \quad \lambda_2 = 2$

$\begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -5v_1 \\ -5v_2 \end{pmatrix}$

$\lambda_1 = -5$
$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow -v_1 + 4v_2 = -5v_1$

$\Rightarrow 4(v_1 + v_2) = 0$

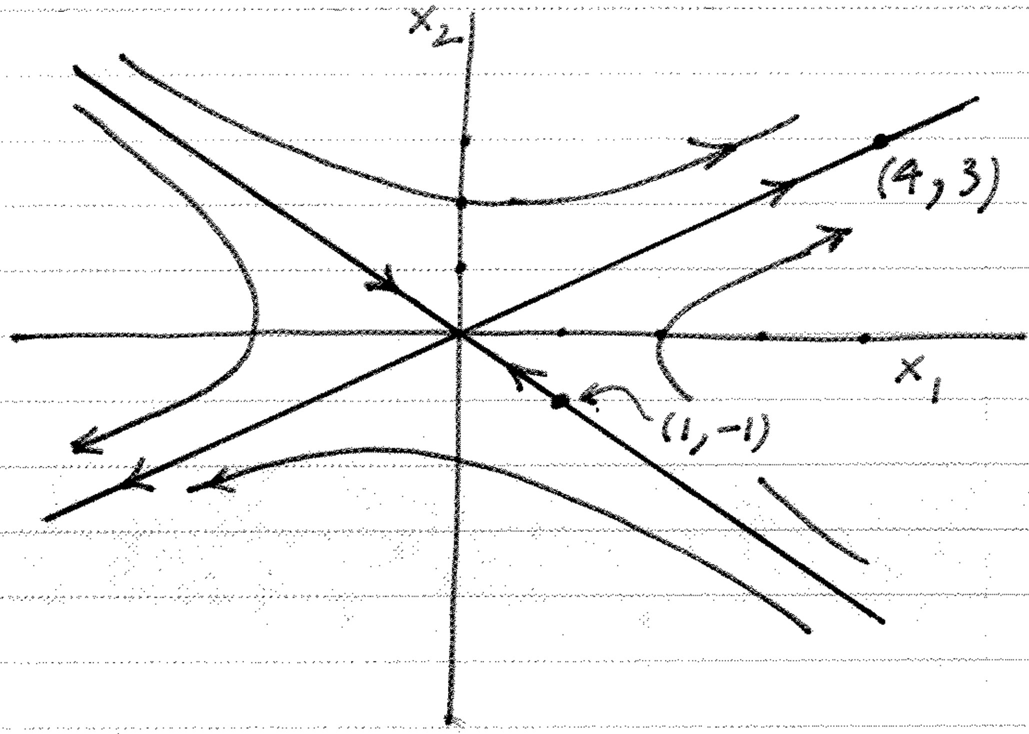
$\Rightarrow v_1 = -v_2$

$\begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}$

$\lambda_2 = 2$
$v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$\Rightarrow -v_1 + 4v_2 = 2v_1$

$\Rightarrow 4v_2 = 3v_1$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = a e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + b e^{2t} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

(i) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & -6 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\boxed{\text{Saddle Point}}$

Char poly $= (\lambda+2)(\lambda+4) - 48$

$= \lambda^2 + 6\lambda - 40 = 0$

$\Rightarrow \lambda = \dfrac{-6 \pm \sqrt{36+160}}{2}$

$= -3 \pm \sqrt{\dfrac{196}{4}} = -3 \pm 7$

$\boxed{\dfrac{196}{4} = 49}$ $\lambda_1 = -10, 4$

$\lambda = -10$

$\begin{pmatrix} -2 & -6 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -10v_1 \\ -10v_2 \end{pmatrix}$ $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\Rightarrow -2v_1 - 6v_2 = -10v_1$

$\Rightarrow 8v_1 = 6v_2$

$\Rightarrow 4v_1 = 3v_2$

$\lambda = 4$

$$\begin{pmatrix} -2 & -6 \\ -8 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \end{pmatrix}$$

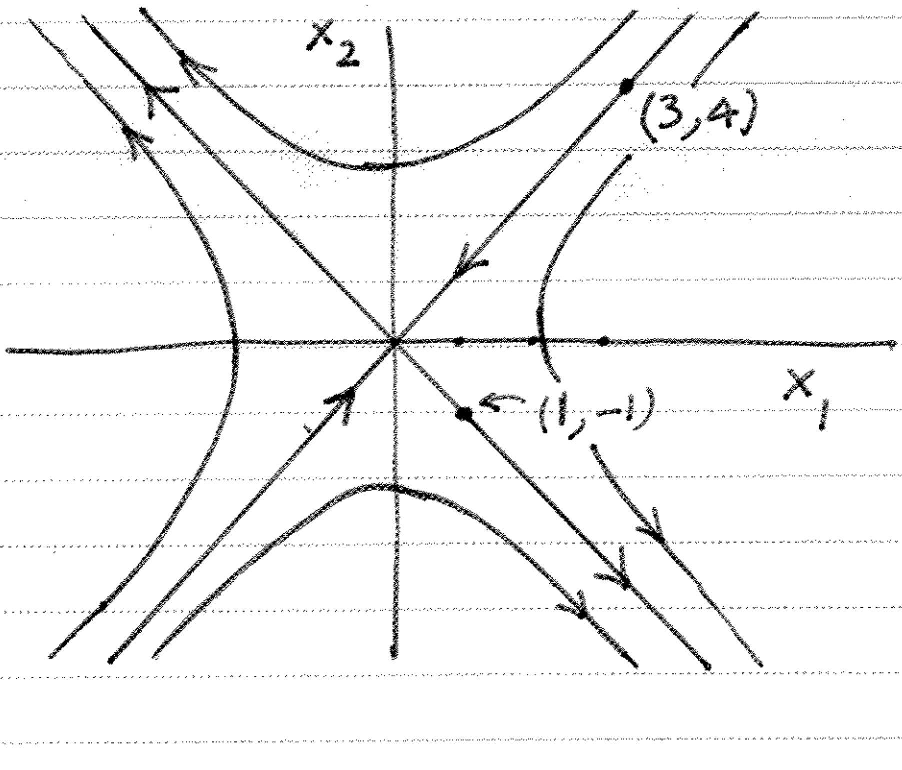
$\Rightarrow -2v_1 - 6v_2 = 4v_1$

$\Rightarrow 6v_1 = -6v_2$

$\Rightarrow v_1 = -v_2$

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = a e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b e^{-10t}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$



(j) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\boxed{\text{Stable Degenerate Node}}$

Eigenvalues are repeated at $-1, -1$.

Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be an eigenvector

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a generalized eigenvector.

$$\begin{pmatrix} -1 & 0 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} ; \begin{pmatrix} -1 & 0 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

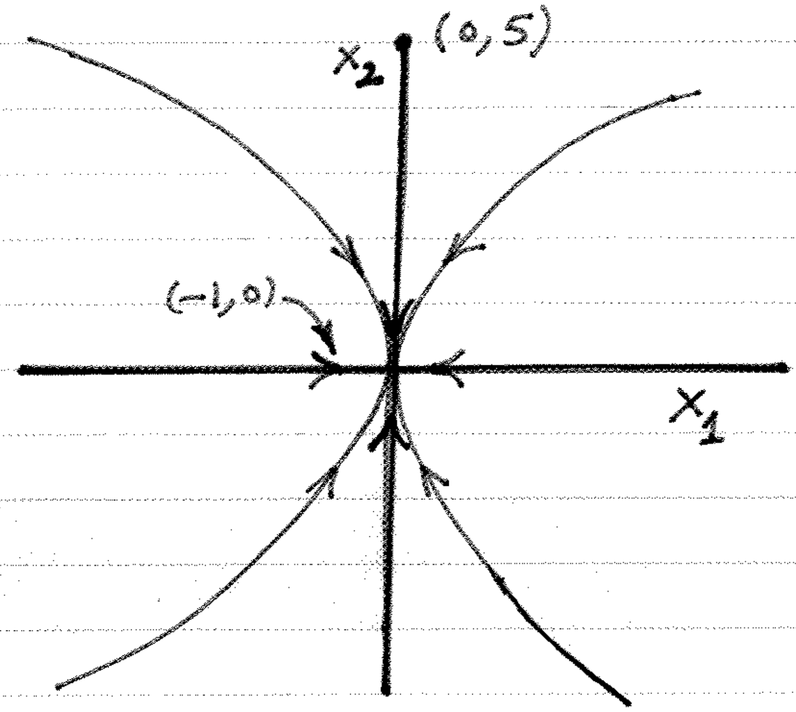
$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = ae^{-t} \begin{pmatrix} 0 \\ 5 \end{pmatrix} + b\left(e^{-t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + te^{-t} \begin{pmatrix} 0 \\ 5 \end{pmatrix} \right)$$

$$= (ae^{-t} + bte^{-t}) \begin{pmatrix} 0 \\ 5 \end{pmatrix} + be^{-t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

For large t,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) \approx bte^{-t} \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$


(0,5)
x_2
(-1,0)
x_1

Type: Saddle Point

(k) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{Char. poly} = (\lambda - 2)^2 - 6$$
$$= \lambda^2 - 4\lambda - 2 = 0$$
$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16 + 8}}{2}$$
$$= 2 \pm \sqrt{\frac{24}{4}}$$
$$= 2 \pm \sqrt{6}$$

$\lambda_1 = 2 + \sqrt{6}$ $\lambda_2 = 2 - \sqrt{6}$

Eigenvector Calculation

$\lambda = \lambda_1$

$$\begin{pmatrix} 2 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 \\ \lambda_1 v_2 \end{pmatrix}$$

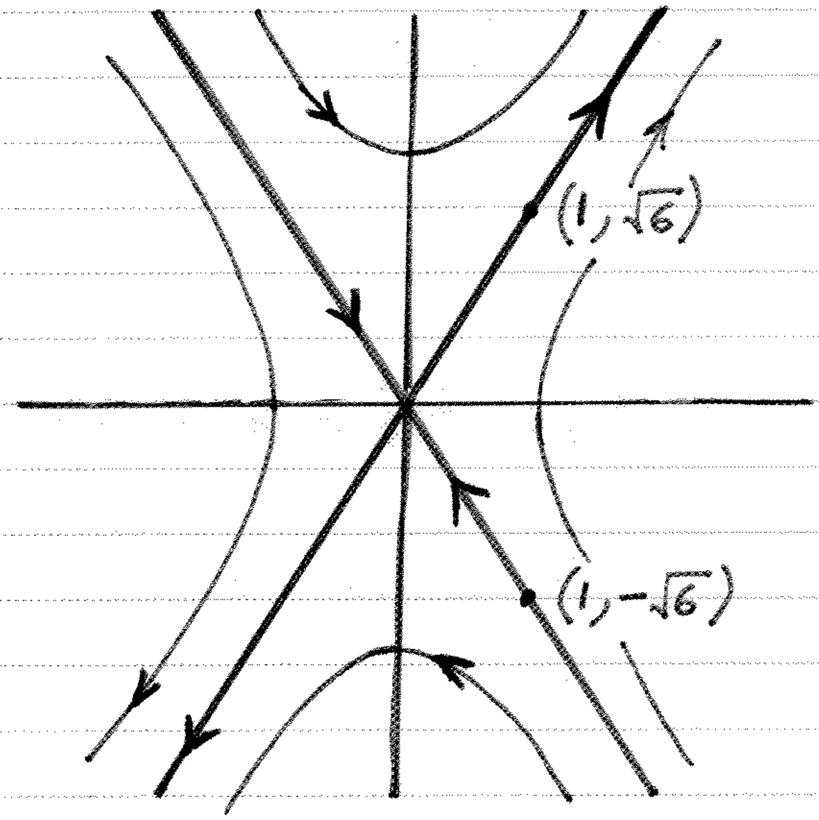
$$2v_1 + v_2 = \lambda_1 v_1$$
$$v_2 = (\lambda_1 - 2) v_1$$
$$\boxed{v_2 = \sqrt{6}\, v_1}$$

$\lambda = \lambda_2$

$$v_2 = -\sqrt{6}\, v_1$$

$\lambda = \lambda_1 > 0 \qquad v = \begin{pmatrix} 1 \\ \sqrt{6} \end{pmatrix}$

$\lambda = \lambda_2 < 0 \qquad v = \begin{pmatrix} 1 \\ -\sqrt{6} \end{pmatrix}$



3 Ans:

(Alternative Sol$^{\underline{n}}$ using matlab)

$$\ddot{x} + b\dot{x} + x = f(t)$$

$$x_1 = x$$
$$x_2 = \dot{x}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -b\,x_2 - x_1 + f(t)$$

$$\dot{X} = AX + b\,f(t)$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix};\ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ X(0) = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$$

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}\,b\,f(\tau)\,d\tau.$$

$$= e^{At}X(0) + e^{At}\int_0^t e^{-A\tau}\,b\,f(\tau)\,d\tau.$$

$$x(t) = c\,e^{At}X(0) + c\,e^{At}\int_0^t e^{-A\tau}\,b\,f(\tau)\,d\tau.$$

where $c = (1 \quad 0)$

A strategy to calculate $x(t)$ is the following :—

① Write down A, b, c and $\underline{x}(0)$

② Calculate

$$c e^{At} \;\&\; e^{-At} b$$

symbolically. If

$$c e^{At} = \begin{pmatrix} \phi_1(t) & \phi_2(t) \end{pmatrix}$$

$$e^{-At} b = \begin{pmatrix} \phi_3(t) \\ \phi_4(t) \end{pmatrix}$$

③ Multiply the forcing function

$$\begin{pmatrix} \phi_3(t) f(t) \\ \phi_4(t) f(t) \end{pmatrix}$$

④ Calculate $\displaystyle\int_0^t \phi_3(\tau) f(\tau)\, d\tau = \psi_3(t)$

$$\int_0^t \phi_4(\tau) f(\tau)\, d\tau = \psi_4(t)$$

(5) Write

$$x(t) = \left(\phi_1(t) \quad \phi_2(t)\right) \mathbf{X}(0) + \phi_1(t)\, \psi_3(t) + \phi_2(t)\, \psi_4(t).$$

(a) $b = 1$, $f(t) = 6$

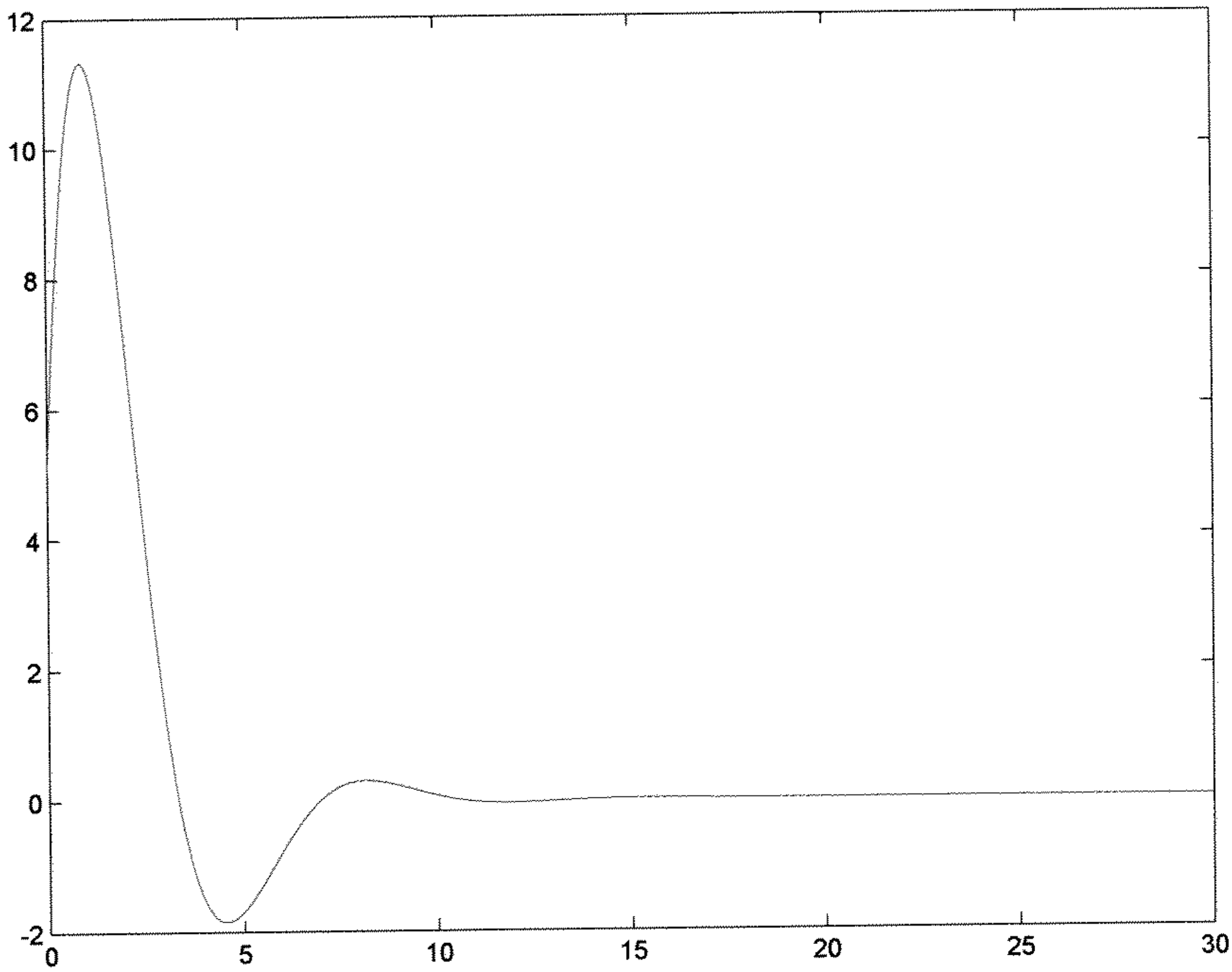
(b) $b = 2$, $f(t) = 6$

(c) $b = 3$, $f(t) = 6$

Note: I have chosen a different $f(t)$ to demonstrate the sol$^{\underline{n}}$ to a step input.

Part (a) Underdamped Case

$x_h(t) = (\Phi_1 \quad \Phi_2)\, \underline{X}(0)$



Exponentially decaying with a large transient.

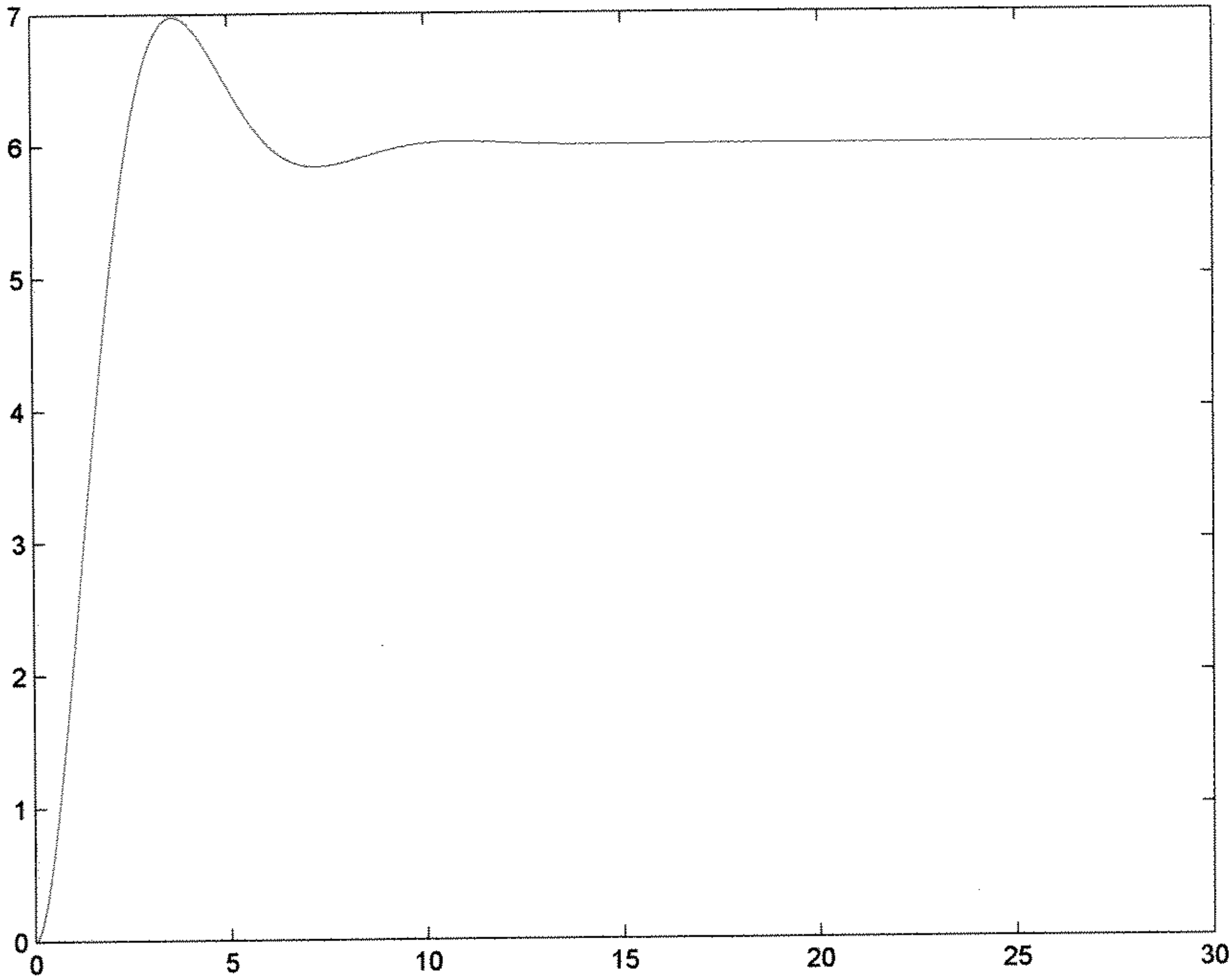
$\ddot{x} + \dot{x} + x = f(t)$

$x(0) = 5 \qquad f(t) = 6, \; t \geq 0$

$\dot{x}(0) = 15$

Part (a) $\boxed{\text{Underdamped case}}$

$$x_p(t) = \phi_1 \psi_3 + \phi_2 \psi_4 .$$



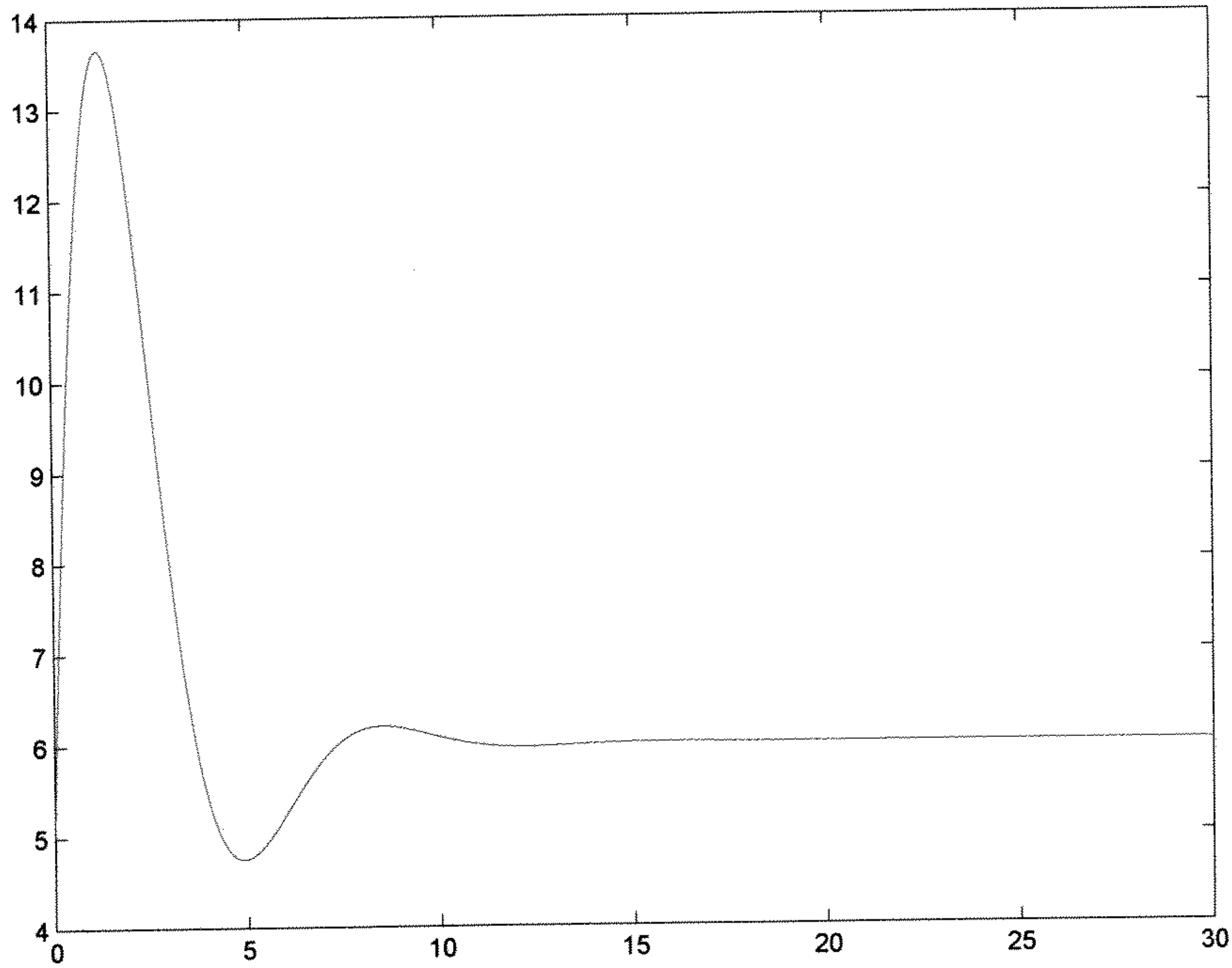
The particular sol$^{\underline{n}}$ is responding to

$f(t) = 6 \quad t \geq 0 .$

$\ddot{x} + \dot{x} + x = f(t)$

part a

$$x(t) = x_h + x_p$$

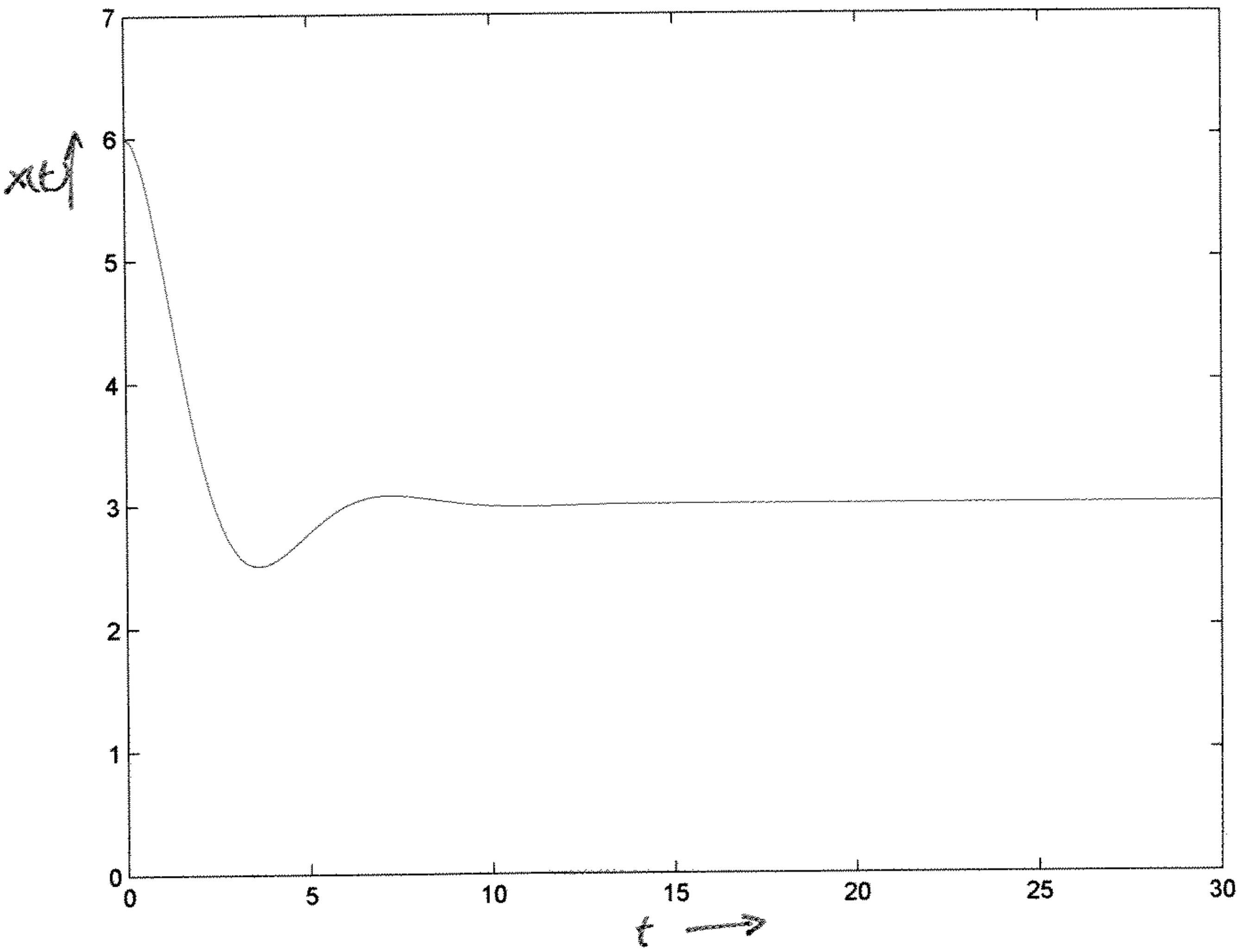


$$\ddot{x} + \dot{x} + x = f(t)$$

$x(0) = 5$ $\quad$ $f(t) = 6 \quad t \geq 0$

$\dot{x}(0) = 15$

part (a)

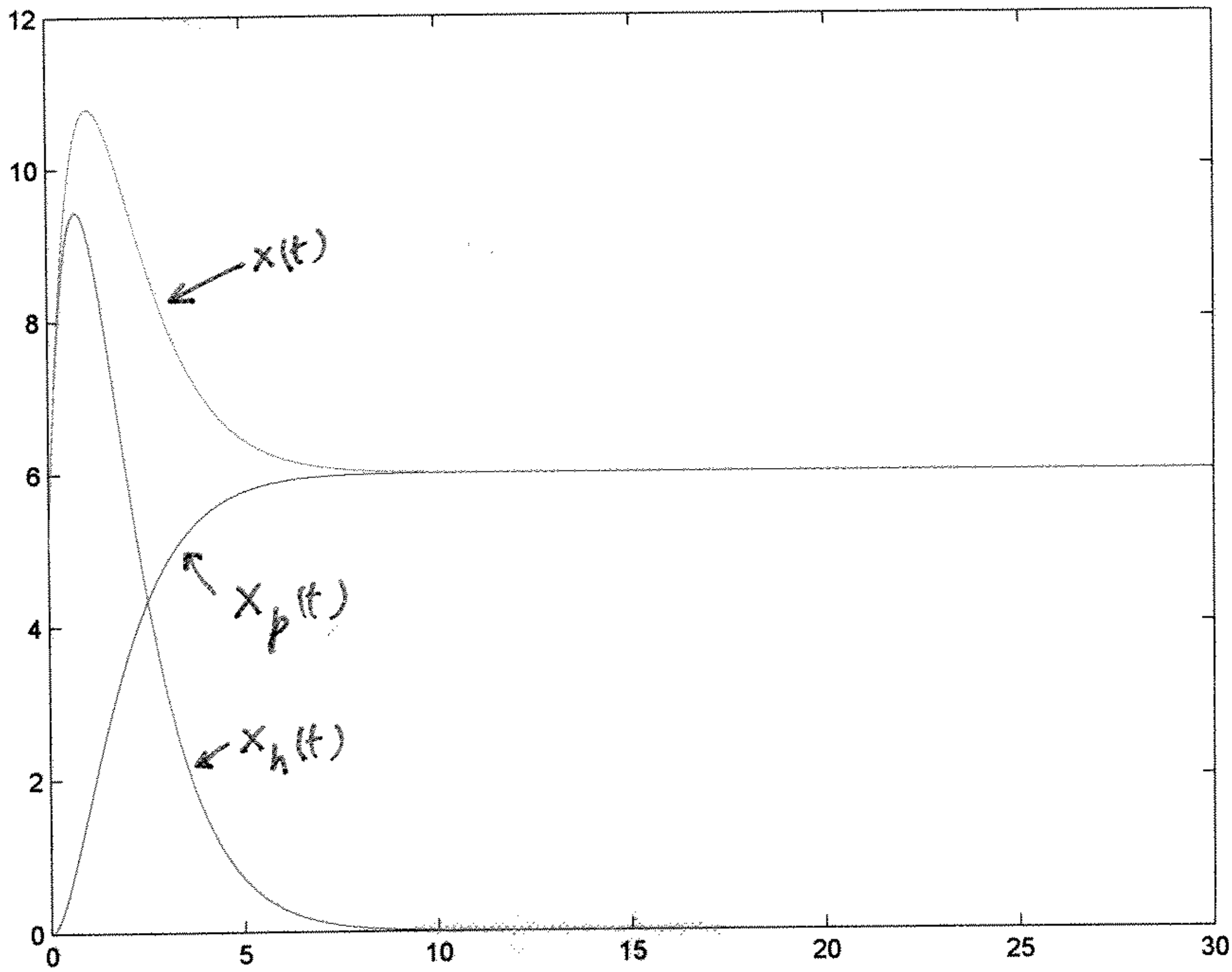


$$\ddot{x} + \dot{x} + x = f(t)$$

$x(0) = 6 \qquad f(t) = 3, \; t \geq 0$

$\dot{x}(0) = 0$

Part b Critically Damped Case



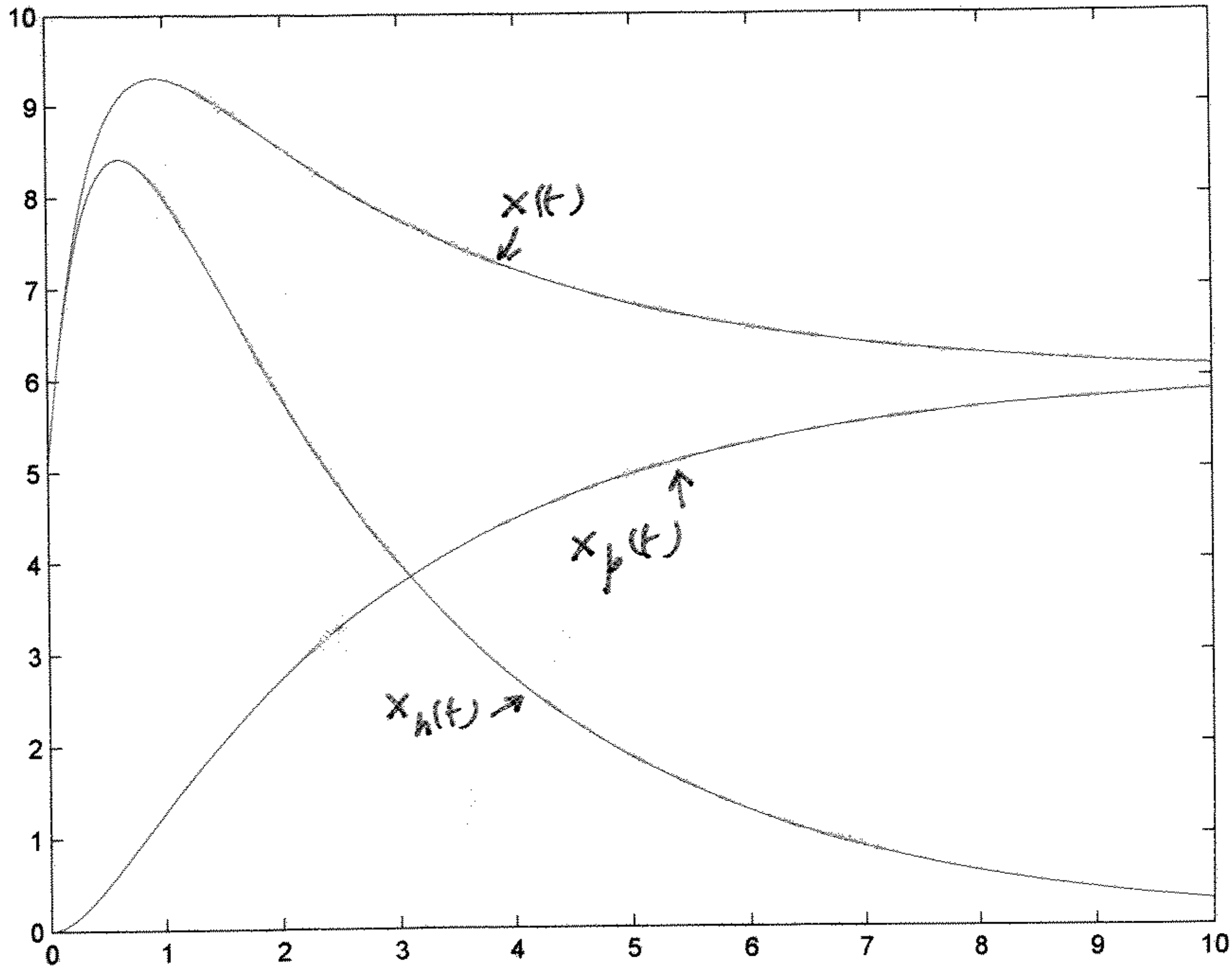
$$\ddot{x} + 2\dot{x} + x = f(t)$$

$$f(t) = 6 \quad t \geq 0$$

$$x(0) = 5$$

$$\dot{x}(0) = 15$$

Part (c) Overdamped case



$$\ddot{x} + 3\dot{x} + x = f(t)$$

$f(t) = 6 \quad t \geq 0$

$x(0) = 5$

$\dot{x}(0) = 15$

H. W. 7

5) Ans:

$$\underline{X}(0) = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} \qquad A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\dot{\underline{X}} = A\underline{X}$$

$$e^{At} = \begin{pmatrix} e^{-2t} & te^{-2t} & \frac{t^2}{2}e^{-2t} \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

$$x_1(t) = 3e^{-2t} + 7te^{-2t} + \frac{11}{2}t^2e^{-2t}$$

$$x_2(t) = 7e^{-2t} + 11te^{-2t}$$

$$x_3(t) = 11e^{-2t}$$

6 Ans!

$$\underline{X}(0) = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}$$

$$\underline{X}(k+1) = A\,\underline{X}(k)$$

$$\Rightarrow \underline{X}(k) = A^k\,\underline{X}(0)$$

$$A = \Lambda + N$$

$$\Lambda = \begin{pmatrix} .5 & & \\ & .5 & \\ & & .5 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^3, N^4, \cdots = 0$$

$$A^k = \Lambda^k + {}^kC_1 \Lambda^{k-1} N + {}^kC_2 \Lambda^{k-2} N^2$$

$$= \begin{pmatrix} .5^k & k(.5)^{k-1} & \frac{k(k-1)}{2}(.5)^{k-2} \\ 0 & .5^k & k(.5)^{k-1} \\ 0 & 0 & .5^k \end{pmatrix}$$

$$x_1(k) = 3\left(\frac{1}{2}\right)^k + 7k\left(\frac{1}{2}\right)^{k-1} + \frac{11}{2}k(k-1)\left(\frac{1}{2}\right)^{k-2}$$

$$x_2(k) = 7\left(\frac{1}{2}\right)^k + 11k\left(\frac{1}{2}\right)^{k-1}$$

$$x_3(k) = 11\left(\frac{1}{2}\right)^k$$

(7) Ans:

$f_0 = 0$ $\qquad\qquad f_j = f_{j-1} + f_{j-2}$.

$f_1 = 1$

$f_2 = 1$

$f_3 = 2$

$f_4 = 3$

$\vdots$

~~$x_1 = f_j$~~ $\quad x_1(j) = f_j$

$x_2(j) = f_{j+1}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(k+1) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(k)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ etc.}$$

$$\lambda_2 \lambda_1^{100} = \lambda_2 \alpha_0 + \alpha_1 \lambda_1 \lambda_2$$

$$\lambda_1 \lambda_2^{100} = \lambda_1 \alpha_0 + \alpha_1 \lambda_1 \lambda_2$$

$$\alpha_0 (\lambda_1 - \lambda_2) = \lambda_1 \lambda_2 \left(\lambda_2^{99} - \lambda_1^{99}\right)$$

$$\alpha_0 = \lambda_1 \lambda_2 \ \frac{\lambda_2^{99} - \lambda_1^{99}}{\lambda_1 - \lambda_2}$$

$$\therefore A^{100} = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 & \alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_0 + \alpha_1 \end{pmatrix}$$

$$\underline{X}(100) = A^{100}\,\underline{X}(0)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{100} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$x_1(100)$ is the 100^{th} term of the fibbonaci sequence.

$$A^{100} = \alpha_0 I + \alpha_1 A$$

$$\lambda_1^{100} = \alpha_0 + \alpha_1 \lambda_1$$

$$\lambda_2^{100} = \alpha_0 + \alpha_1 \lambda_2$$

$$\alpha_1 (\lambda_1 - \lambda_2) = \lambda_1^{100} - \lambda_2^{100}$$

$$\alpha_1 = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2}$$

$$\alpha_0 + \alpha_1 =$$

$$\frac{\lambda_1 \lambda_2^{100} - \lambda_2 \lambda_1^{100} + \lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2}$$

$$\underline{X}(100) = \begin{pmatrix} \alpha_1 \\ \alpha_0 + \alpha_1 \end{pmatrix}$$

$$x_1(100) = \alpha_1$$

$$\therefore \boxed{f_{100} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2}}$$

λ_1 & λ_2 are roots of.

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda-1 \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1+\sqrt{5}}{2} \qquad \frac{1-\sqrt{5}}{2}$$

$$\lambda_1 = \qquad \lambda_2 =$$

$$\lambda_1 - \lambda_2 = \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2} = \sqrt{5}$$

$$f_{100} = \frac{1}{\sqrt{5}}\left(\lambda_1^{100} - \lambda_2^{100}\right)$$

Note that since $|\lambda_2| < 1$

$$\lambda_2^{100} \approx 0.$$

$$f_{100} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{100}.$$

8 Ans:

$$S = I + A + A^2 + \cdots + A^{100}.$$

$$SA = A + A^2 + \cdots + A^{100} + A^{101}$$

$$S(I - A) = I - A^{101}$$

$$\boxed{S = (I - A^{101})(I - A)^{-1}}$$

If

$$S_N = I + A + A^2 + \cdots + A^N.$$

$$S_N = (I - A^{N+1})(I - A)^{-1}.$$

$\because$ eigenvalues of A all have magnitude < 1 we have

$$\underset{N \to \infty}{Lt} S_N = (I - A)^{-1}.$$

$$A^{101} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 .$$

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^{101} \\ \lambda_2^{101} \\ \lambda_3^{101} \end{pmatrix}$$

$$\lambda_1 = -\frac{1}{2} , \ \lambda_2 = -\frac{1}{4}, \ \lambda_3 = -\frac{3}{4}$$

$\lambda_1 \quad \lambda_2 \quad \lambda_3$ Eigenvalues and

$v_1 \quad v_2 \quad v_3$ Eigenvectors.

$$P = [v_1 \ v_2 \ v_3]$$

$$P^{-1} A P = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \Lambda$$

$$I + \Lambda + \cdots + \Lambda^{100} =$$

$$\begin{pmatrix} \dfrac{1-\lambda_1^{101}}{1-\lambda_1} & 0 & 0 \\ 0 & \dfrac{1-\lambda_2^{101}}{1-\lambda_2} & 0 \\ 0 & 0 & \dfrac{1-\lambda_3^{101}}{1-\lambda_3} \end{pmatrix}$$

$$I + A + \cdots + A^{100} =$$

$$P \begin{pmatrix} \ \ \end{pmatrix} P^{-1}$$