Home Work 6 (Answers)

1

(a) $$A = \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix} \qquad A = \Lambda + N$$

$$\Lambda = \begin{pmatrix} \lambda_0 & & & 0 \\ & \lambda_0 & & \\ & & \lambda_0 & \\ 0 & & & \lambda_0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Check that Λ and N commute ie

$$\Lambda N = N \Lambda$$

$$e^{At} = e^{\Lambda t} e^{Nt}$$

Remark:

$$e^{(\Lambda + N)t} = e^{\Lambda t} e^{Nt}$$

because Λ & N commute

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_0 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_0 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_0 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_0 t} \end{pmatrix}$$

check that $N^4 = 0$, $N^5 = 0$ ....

In fact $N^j = 0$ for all $j \geqslant 4$.

$$e^{Nt} = I + Nt + \frac{N^2 t^2}{2!} + \frac{N^3 t^3}{3!}$$

6.3

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Nt = \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{N^2 t^2}{2!} = \begin{pmatrix} 0 & 0 & t^2/2! & 0 \\ 0 & 0 & 0 & t^2/2! \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{N^3 t^3}{3!} = \begin{pmatrix} 0 & 0 & 0 & t^3/3! \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

6.4

$$e^{Nt} = \begin{pmatrix} 1 & t & t^2/2! & t^3/3! \\ 0 & 1 & t & t^2/2! \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{\lambda_0 t} & te^{\lambda_0 t} & \frac{t^2}{2!}e^{\lambda_0 t} & \frac{t^3}{3!}e^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} & te^{\lambda_0 t} & \frac{t^2}{2!}e^{\lambda_0 t} \\ 0 & 0 & e^{\lambda_0 t} & te^{\lambda_0 t} \\ 0 & 0 & 0 & e^{\lambda_0 t} \end{pmatrix}$$

(b) $$B = \begin{pmatrix} \lambda_0 & 0 & 1 & 0 \\ 0 & \lambda_0 & 0 & 1 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}$$

Λ: as in part (a)

$$N_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \boxed{B = \Lambda + N_1}$$

Check that Λ & N_1 commutes.

$$e^{Bt} = e^{(\Lambda + N_1)t} = e^{\Lambda t} e^{N_1 t}$$

$e^{\Lambda t}$ is on page 6.2.

$$N_1^2 = 0,\ N_1^3 = 0,\ N_1^4 = 0 \ \ldots\ N_1^j = 0,\ j \geq 2$$

6.6

$$e^{N_1 t} = I + N_1 t$$

$$= \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{Bt} = \begin{pmatrix} e^{\lambda_0 t} & 0 & te^{\lambda_0 t} & 0 \\ 0 & e^{\lambda_0 t} & 0 & te^{\lambda_0 t} \\ 0 & 0 & e^{\lambda_0 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_0 t} \end{pmatrix}$$

c) $$C = \begin{pmatrix} \lambda_0 & 0 & 0 & 1 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}$$

Λ : as in parts a and b

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_2^j = 0 \quad j \geqslant 2.$$

$$e^{N_2 t} = I + N_2 t$$

$$= \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6.8

$$e^{Ct} = \begin{pmatrix} e^{\lambda_0 t} & 0 & 0 & te^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_0 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_0 t} \end{pmatrix}$$

(d) $D = \begin{pmatrix} \lambda_0 & 0 & 1 & 1 \\ 0 & \lambda_0 & 0 & 1 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}$

Λ: as in parts (a), (b) & (c)

$$N_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$N_3^2 = 0 \quad N_3^j = 0 \quad j \geq 2.$

6.9

$$e^{N_3 t} = I + N_3 t$$

$$= \begin{pmatrix} 1 & 0 & t & t \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{Dt} = \begin{pmatrix} e^{\lambda_0 t} & 0 & te^{\lambda_0 t} & te^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} & 0 & te^{\lambda_0 t} \\ 0 & 0 & e^{\lambda_0 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_0 t} \end{pmatrix}$$

(e) $E = \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 1 & \lambda_0 & 1 & 0 \\ 0 & 1 & \lambda_0 & 1 \\ 0 & 0 & 1 & \lambda_0 \end{pmatrix}$

$\Lambda =$ Same as (a), (b), (c), (d)

$N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Λ & N_4 commutes, so we will have

$e^{Et} = e^{\Lambda t} e^{N_4 t}$ (as before)

However $e^{N_4 t}$ is not easy to compute by hand. It is not a nilpotent matrix ie $N_4^j \neq 0$ for any j.

(6.11)

```
>> N4=[0 1 0 0;1 0 1 0;0 1 0 1;0 0 1 0]

N4 =
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
>> [v1 v2]=eig(N4)

v1 =
```

$$\begin{pmatrix} 0.3717 & -0.6015 & -0.6015 & 0.3717 \\ -0.6015 & 0.3717 & -0.3717 & 0.6015 \\ 0.6015 & 0.3717 & 0.3717 & 0.6015 \\ -0.3717 & -0.6015 & 0.6015 & 0.3717 \end{pmatrix} = P$$

```
v2 =
```

$$\begin{matrix} -1.6180 & 0 & 0 & 0 \\ 0 & -0.6180 & 0 & 0 \\ 0 & 0 & 0.6180 & 0 \\ 0 & 0 & 0 & 1.6180 \end{matrix}$$

$0.6180 = \frac{\sqrt{5}-1}{2}$

$1.6180 = \frac{\sqrt{5}+1}{2}$

The four eigenvalues of N_4 are at

$$\frac{\pm\sqrt{5}\pm 1}{2}$$

ie

$$\frac{\sqrt{5}+1}{2}, \quad \frac{\sqrt{5}-1}{2}, \quad \frac{-\sqrt{5}+1}{2}, \quad \frac{-\sqrt{5}-1}{2}$$

$$P^{-1} N_4 P = \begin{pmatrix} -\lambda_1 & & & \\ & -\lambda_2 & & \\ & & \lambda_2 & \\ & & & \lambda_1 \end{pmatrix}$$

$$\lambda_1 = \frac{\sqrt{5}+1}{2}, \quad \lambda_2 = \frac{\sqrt{5}-1}{2}$$

$$P^{-1} e^{N_4 t} P = \begin{pmatrix} e^{-\lambda_1 t} & & & \\ & e^{-\lambda_2 t} & & \\ & & e^{+\lambda_2 t} & \\ & & & e^{+\lambda_1 t} \end{pmatrix}$$

$$e^{N_4 t} = P \begin{pmatrix} e^{-\lambda_1 t} & & & \\ & e^{-\lambda_2 t} & & \\ & & e^{\lambda_2 t} & \\ & & & e^{\lambda_1 t} \end{pmatrix} P^{-1}$$

$$e^{Et} = e^{\Lambda t} e^{N_4 t} =$$

$$\begin{pmatrix} e^{\lambda_0 t} & & & \\ & e^{\lambda_0 t} & & 0 \\ & & e^{\lambda_0 t} & \\ 0 & & & e^{\lambda_0 t} \end{pmatrix} \cdot e^{N_4 t}$$

This multiplication is same as scaling every element of $e^{N_4 t}$ by $e^{\lambda_0 t}$.

$$\therefore e^{Et} = P \begin{pmatrix} e^{(\lambda_0 - \lambda_1)t} & & & \\ & e^{(\lambda_0 - \lambda_2)t} & & 0 \\ & & e^{(\lambda_0 + \lambda_2)t} & \\ 0 & & & e^{(\lambda_0 + \lambda_1)t} \end{pmatrix} P^{-1}$$

Remark:

The matrix P of the eigenvectors of N_4 were computed using matlab. on page 6.11. It is not too hard to show that for an eigenvalue λ of N_4, the corresponding eigen-vector is given by

$$\begin{pmatrix} \lambda \\ \lambda^2 \\ \lambda(\lambda^2-1) \\ \lambda^2-1 \end{pmatrix}$$

Since the eigenvalue are known and given on page 6.11, the eigenvectors are easily written by hand without using matlab.

6.15

(2) Ans:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$$

Char poly $p(\lambda)$ of A is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6$$

$$= (\lambda-1)(\lambda-2)(\lambda-3).$$

Eigenvalues at 1, 2, 3.

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

To find $\alpha_0, \alpha_1, \alpha_2$ we substitute eigenvalues for A and obtain

$$\boxed{\begin{aligned} e^{t} &= \alpha_0 + \alpha_1 + \alpha_2 \\ e^{2t} &= \alpha_0 + 2\alpha_1 + 4\alpha_2 \\ e^{3t} &= \alpha_0 + 3\alpha_1 + 9\alpha_2 \end{aligned}}$$

6.16

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} e^{t} \\ e^{2t} \\ e^{3t} \end{pmatrix}$$

Solve for $\alpha_0, \alpha_1, \alpha_2$ and substitute back.

See attached pages 6.17 & 6.18

6.17

```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> M=[1 1 1;1 2 4;1 3 9]

M =

     1     1     1
     1     2     4
     1     3     9

>> adjM=round(inv(M)*det(M))

adjM =

     6    -6     2
    -5     8    -3
     1    -2     1

>> m=det(M)

m =

     2

>> syms t
>> b=[exp(t);exp(2*t);exp(3*t)]

b =

[   exp(t)]
[ exp(2*t)]
[ exp(3*t)]

>> alpha=adjM*b

alpha =

[  6*exp(t)-6*exp(2*t)+2*exp(3*t)]
[ -5*exp(t)+8*exp(2*t)-3*exp(3*t)]
[      exp(t)-2*exp(2*t)+exp(3*t)]

>>
```

$$\boxed{\begin{aligned}\alpha_0 &= 3e^{t} - 3e^{2t} + e^{3t}\\ \alpha_1 &= -\frac{5}{2}e^{t} + 4e^{2t} - \frac{3}{2}e^{3t}\\ \alpha_2 &= \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}\end{aligned}}$$

obtained by dividing with det M.

```
alpha0=alpha(1)/2

alpha0 =

3*exp(t)-3*exp(2*t)+exp(3*t)

>> alpha1=alpha(2)/2

alpha1 =

-5/2*exp(t)+4*exp(2*t)-3/2*exp(3*t)

>> alpha2=alpha(3)/2

alpha2 =

1/2*exp(t)-exp(2*t)+1/2*exp(3*t)

>> A=[0 1 0;0 0 1;6 -11 6]

A =

     0     1     0
     0     0     1
     6   -11     6

>> B=A*A

B =

     0     0     1
     6   -11     6
    36   -60    25

>> I=[1 0 0;0 1 0;0 0 1];
>> exp=alpha0*I+alpha1*A+alpha2*B

exp =

[        3*exp(t)-3*exp(2*t)+exp(3*t),   -5/2*exp(t)+4*exp(2*t)-3/2*exp(3*t),
1/2*exp(t)-exp(2*t)+1/2*exp(3*t)]
[      3*exp(t)-6*exp(2*t)+3*exp(3*t),   -5/2*exp(t)+8*exp(2*t)-9/2*exp(3*t),
1/2*exp(t)-2*exp(2*t)+3/2*exp(3*t)]
[     3*exp(t)-12*exp(2*t)+9*exp(3*t), -5/2*exp(t)+16*exp(2*t)-27/2*exp(3*t),
1/2*exp(t)-4*exp(2*t)+9/2*exp(3*t)]
```

Alternatively, the eigenvectors of A are at $\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$ for eigenvalue λ.

Hence $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$P^{-1}e^{At}P = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$e^{At} = P \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} P^{-1}$$

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```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> P=[1 1 1;1 2 3;1 4 9]

P =

     1     1     1
     1     2     3
     1     4     9

>> syms t
>> diag=[exp(t) 0 0;0 exp(2*t) 0; 0 0 exp(3*t)]

diag =

[   exp(t),         0,         0]
[        0,  exp(2*t),         0]
[        0,         0,  exp(3*t)]

>> exp=P*diag*inv(P)

exp =

[          3*exp(t)-3*exp(2*t)+exp(3*t),   -5/2*exp(t)+4*exp(2*t)-3/2*exp(3*t),       1/2* 
exp(t)-exp(2*t)+1/2*exp(3*t)]
[        3*exp(t)-6*exp(2*t)+3*exp(3*t),   -5/2*exp(t)+8*exp(2*t)-9/2*exp(3*t),     1/2*ex 
p(t)-2*exp(2*t)+3/2*exp(3*t)]
[       3*exp(t)-12*exp(2*t)+9*exp(3*t), -5/2*exp(t)+16*exp(2*t)-27/2*exp(3*t),     1/2*ex 
p(t)-4*exp(2*t)+9/2*exp(3*t)]

>>
```

$e^{At} =$

Notice that the results match with page 6·18

6.21

Let $B = A^2$, it follows that if v is an eigenvector of A with eigenvalue λ.

$$Av = \lambda v$$

then

$$Bv = A^2 v = \lambda^2 v$$

Hence λ^2 is an eigenvalue of B with eigenvector v.

Thus $\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$ is an eigenvector of B

with eigenvalue λ^2 where λ is an eigenvalue of A.

$\therefore \lambda = 1, 2, 3.$

$\lambda^2 = 1, 4, 9.$

B has eigenvalues at $1, 4, 9$ with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

respectively.

e^{Bt} can now be easily computed.

Proceeding as before

$$e^{Bt} = \alpha_0 I + \alpha_1 B + \alpha_2 B^2$$

Since 1, 4, 9 are eigenvalues of B we have

$$e^{t} = \alpha_0 + \alpha_1 + \alpha_2$$

$$e^{4t} = \alpha_0 + 4\alpha_1 + 16\alpha_2$$

$$e^{9t} = \alpha_0 + 9\alpha_1 + 81\alpha_2$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 16 \\ 1 & 9 & 81 \end{pmatrix}}_{Q} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} e^{t} \\ e^{4t} \\ e^{9t} \end{pmatrix}$$

e^{Bt} has been calculated in the next two pages.

Using Toolbox Path Cache. Type "help toolbox_path_cache" for more info.

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To get started, select "MATLAB Help" from the Help menu.

Calculation of e^{Bt}

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 16 \\ 1 & 9 & 81 \end{pmatrix} = Q$$

$$\begin{pmatrix} 180 & -72 & 12 \\ -65 & 80 & -15 \\ 5 & -8 & 3 \end{pmatrix} = \operatorname{adj} Q$$

120 ← det Q

```
>> syms t
>>
>> b=[exp(t);exp(4*t);exp(9*t)]

b =

[   exp(t)]
[ exp(4*t)]
[ exp(9*t)]
```

$$Q^{-1}\begin{pmatrix} e^{t} \\ e^{4t} \\ e^{9t} \end{pmatrix}$$

```
alpha =

[   3/2*exp(t)-3/5*exp(4*t)+1/10*exp(9*t)]
[ -13/24*exp(t)+2/3*exp(4*t)-1/8*exp(9*t)]
[ 1/24*exp(t)-1/15*exp(4*t)+1/40*exp(9*t)]
```

α_0, α_1, α_2

$$\begin{pmatrix} 0 & 0 & 1 \\ 6 & -11 & 6 \\ 36 & -60 & 25 \end{pmatrix} = B$$

```
>> I=[1 0 0;0 1 0;0 0 1]

I =

     1     0     0
     0     1     0
```

```
     0         0         1
```

6.25

```
exp =

[          3*exp(t)-3*exp(4*t)+exp(9*t),    -5/2*exp(t)+4*exp(4*t)-3/2*exp(9*t),        1/2* ↙
exp(t)-exp(4*t)+1/2*exp(9*t)]
[        3*exp(t)-6*exp(4*t)+3*exp(9*t),    -5/2*exp(t)+8*exp(4*t)-9/2*exp(9*t),     1/2*ex ↙
p(t)-2*exp(4*t)+3/2*exp(9*t)]
[       3*exp(t)-12*exp(4*t)+9*exp(9*t),  -5/2*exp(t)+16*exp(4*t)-27/2*exp(9*t),     1/2*ex ↙
p(t)-4*exp(4*t)+9/2*exp(9*t)]

>>
```

e^{Bt}

2 (b)

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```
Type "help toolbox_path_cache" for more info.

To get started, select "MATLAB Help" from the Help menu.

>> A=[0 1 0;0 0 1;6 -11 6]

A =

     0     1     0
     0     0     1
     6   -11     6
```

Verify that A, e^A, e^{A^2} commute

```
>> AA=A*A

AA =

     0     0     1
     6   -11     6
    36   -60    25
```

$= A^2$

```
>> B=expm(A)

B =

    6.0732   -7.3678    4.0129
   24.0771  -38.0682   16.7093
  100.2560 -159.7256   62.1878
```

$= e^A$

```
>> C=expm(AA)

C =

  1.0e+005 *

    0.0795   -0.1194    0.0400
    0.2399   -0.3603    0.1205
    0.7228   -1.0852    0.3625
```

$= e^{A^2}$

```
>> round(A*B-B*A)

ans =

     0     0     0
     0     0     0
     0     0     0

>> round(A*C-C*A)

ans =

     0     0     0
     0     0     0
     0     0     0
```

Verifying that the matrices commute.

```
>> round(B*C-C*B)

ans =

     0     0     0
     0     0     0
     0     0     0

>> A*B-B*A

ans =

  1.0e-011 *

    0.0007   -0.0071   -0.0014
   -0.0256    0.0028   -0.0249
    0.0455   -0.2160   -0.0028

>> A*C-C*A

ans =

  1.0e-008 *

    0.0084    0.0102   -0.0087
    0.0626   -0.0233   -0.0044
    0.2474   -0.1455    0.0146

>> B*C-C*B

ans =

  1.0e-007 *

    0.0128    0.0079    0.0028
    0.0594   -0.0023    0.0192
    0.0466    0.1770   -0.0093
```

4 Ans:

Char. poly of M is

$$\lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24$$

$$= (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4).$$

can be checked by multiplication.

Let λ be an eigenvalue of M with eigenvector $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = v$.

Writing $Mv = \lambda v$ we have

$v_2 = \lambda v_1$, $v_3 = \lambda v_2$, $v_4 = \lambda v_3$.

& $-24 v_1 + 50 v_2 - 35 v_3 + 10 v_4 = \lambda v_4$.

It follows that

$$v_2 = \lambda v_1$$
$$v_3 = \lambda^2 v_1$$
$$v_4 = \lambda^3 v_1.$$

&

$$-24 v_1 + 50 \lambda v_1 - 35\lambda^2 v_1 + 10\lambda^3 v_1 = \lambda^4 v_1$$

$$\Rightarrow \underbrace{(\lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24)}_{}v_1 = 0$$

λ satisfies this eq^n already since λ is an eigenvalue

$$v = \begin{pmatrix} v_1 \\ \lambda v_1 \\ \lambda^2 v_1 \\ \lambda^3 v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{pmatrix} v_1$$

Note that

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}$$

can be written by inspection.

Using Matlab to verify that $P^{-1}MP$ is diagonal.

```
>> P=[1 1 1 1;1 2 3 4;1 4 9 16;1 8 27 64]

P =

     1     1     1     1
     1     2     3     4
     1     4     9    16
     1     8    27    64

>> M=[0 1 0 0;0 0 1 0;0 0 0 1;-24 50 -35 10]

M =

     0     1     0     0
     0     0     1     0
     0     0     0     1
   -24    50   -35    10

>> inv(P)*M*P

ans =

    1.0000         0    0.0000    0.0000
   -0.0000    2.0000   -0.0000   -0.0000
   -0.0000         0    3.0000    0.0000
   -0.0000   -0.0000   -0.0000    4.0000
```

$= P^{-1}MP$

3) Ans

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```
Using Toolb...                    e "help toolbox_path_cache" for more info.

To get ...                   AB Help" from the Help menu.

>> M=[45 120 315 672;-60 -252 -756 -1680;45 216 651 1440;-12 -60 -180 -396]

M =

     45      120      315      672
    -60     -252     -756    -1680
     45      216      651     1440
    -12      -60     -180     -396

>> [v1 v2]=jordan(M)

v1 =

  -67.8000  -11.6000 -100.8000  -12.6000
  -60.0000         0         0         0
  117.0000    1.0000   72.0000    1.0000
  -40.8000         0  -28.8000         0
```

Eigenvectors.

generalized Eigenvectors.

u_1 u_2 u_3 u_4

```
v2 =

    12     1     0     0
     0    12     0     0
     0     0    12     1
     0     0     0    12
```

Eigenvalues are at 12, 12, 12, 12

```
>> M*v1(:,1)-12*v1(:,1)

ans =

  1.0e-011 *

    0.1364
         0
    0.7276
   -0.2217
```

$Mu_1 = 12u_1$

Verification

```
>> M*v1(:,3)-12*v1(:,3)

ans =

  1.0e-011 *

   -0.2274
         0
         0
    0.1478
```

$Mu_3 = 12u_3$

```
>> M*v1(:,2)-12*v1(:,2)-v1(:,1)

ans =
```

$Mu_2 = 12u_2 + u_1$

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```
  1.0e-013 *

   -0.1421
         0
         0
   -0.1421

>> M*v1(:,4)-12*v1(:,4)-v1(:,3)

ans =

  1.0e-013 *

   -0.1421
         0
         0
   -0.1066

>>
```

$$Mu_4 = 12u_4 + u_3$$

3(b)

$$e^{Mt} = \alpha_0 I + \alpha_1 M + \alpha_2 M^2 + \alpha_3 M^3$$

$\lambda = 12$ is repeated 4 times.. We obtain

$$\boxed{\begin{aligned} e^{\lambda t} &= \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \alpha_3 \lambda^3 \\ t e^{\lambda t} &= 0 + \alpha_1 + 2\alpha_2 \lambda + 3\alpha_3 \lambda^2 \\ t^2 e^{\lambda t} &= 0 + 0 + 2\alpha_2 + 6\alpha_3 \lambda \\ t^3 e^{\lambda t} &= 0 + 0 + 0 + 6\alpha_3 \end{aligned}}$$

taking $\frac{d}{d\lambda}$

Substitute $\lambda = 12$ and solve for $\alpha_0, \alpha_1, \alpha_2, \alpha_3$

5 (Ans)

Cache. Type "help toolbox_path_cache" for more info.

lect "MATLAB Help" from the Help menu.

```
>> A=[2 4 6 8 9 3 1 2 3 4;3 5 1 7 5 3 2 1 7 4;3 4 6 7 3 2 1 2 3 4;5 6 7 8 9 0 9  8 7 6;
5 5 4 3 3 2 2 3 3 4;4 5 6 7 7 8 9 0  8 7 ;6 6 5 4 5 3 4 5  7 6; 5 4 4 3 3 3 4 5  6 0; 9 1
 2 3 2 1  3 4 5 6; 7 8 9 0  7 6 5 4 4 3; 3 3 3 4 4 5 4 5 5 5]

A =

     2     4     6     8     9     3     1     2     3     4
     3     5     1     7     5     3     2     1     7     4
     3     4     6     7     3     2     1     2     3     4
     5     6     7     8     9     0     9     8     7     6
     5     5     4     3     3     2     2     3     3     4
     4     5     6     7     7     8     9     0     8     7
     6     6     5     4     5     3     4     5     7     6
     5     4     4     3     3     3     4     5     6     0
     9     1     2     3     2     1     3     4     5     6
     7     8     9     0     7     6     5     4     4     3
     3     3     3     4     4     5     4     5     5     5
```

An arbitrary 10x10 matrix.

```
>> A=transpose(A)*A

A =

   288   240   252   219   254   162   220   202   278   236
   240   269   274   244   289   179   222   185   274   224
   252   274   309   257   310   187   238   197   272   236
   219   244   257   334   309   165   223   177   296   259
   254   289   310   309   357   198   261   206   310   266
   162   179   187   165   198   170   164   102   199   163
   220   222   238   223   261   164   254   176   268   218
   202   185   197   177   206   102   176   189   210   171
   278   274   272   296   310   199   268   210   340   271
   236   224   236   259   266   163   218   171   271   255
```

$A^T A$ is a positive definite matrix.

```
>> eig(A)

ans =

  1.0e+003 *

    0.0013
    0.0104
    0.0131
    0.0133
    0.0271
    0.0479
    0.0718
    0.0878
    0.1159
    2.3764
```

Eigenvalues are all positive.
A is positive definite.

```
>> B=round(A/10)

B =
```

6·33

```
    29    24    25    22    25    16    22    20    28    24
    24    27    27    24    29    18    22    19    27    22
    25    27    31    26    31    19    24    20    27    24
    22    24    26    33    31    17    22    18    30    26
    25    29    31    31    36    20    26    21    31    27
    16    18    19    17    20    17    16    10    20    16
    22    22    24    22    26    16    25    18    27    22
    20    19    20    18    21    10    18    19    21    17
    28    27    27    30    31    20    27    21    34    27
    24    22    24    26    27    16    22    17    27    26

>> eig(B)

ans =

   -0.4568
    0.6346
    0.7670
    1.7959
    3.5152
    4.8949
    7.2713
    9.0223
   11.5502
  238.0054
```

By scaling and rounding we lose positive definiteness.

The ten eigenvectors of A are orthonormal.

```
>> [v1 v2]=eig(A)

v1 =

    0.2583   -0.4412   -0.2624    0.0482   -0.1113    0.4477   -0.1022   -0.2214    0.544
1    0.3149
    0.3095    0.5878   -0.3319    0.1990    0.3998    0.1464    0.1583    0.3036    0.079
9    0.3226
   -0.4420   -0.0212   -0.1256   -0.4958   -0.2891    0.0955    0.3663    0.4382    0.088
4    0.3403
    0.3188   -0.1287   -0.0785   -0.3962    0.1054    0.1273    0.1398   -0.3591   -0.655
8    0.3357
   -0.0448   -0.4029    0.1357    0.6655   -0.0755   -0.1384    0.2976    0.2241   -0.268
8    0.3720
    0.2131   -0.1057    0.5288   -0.1811    0.1289    0.1556   -0.5409    0.4941   -0.048
9    0.2260
    0.2687    0.0101   -0.2606   -0.1173   -0.2735   -0.7698   -0.2684    0.0169    0.125
0    0.3011
    0.0907    0.1215    0.6156   -0.1277    0.2006   -0.2210    0.4216   -0.3118    0.398
2    0.2432
   -0.6269   -0.0397   -0.1205    0.0395    0.5012   -0.0990   -0.3603   -0.2553    0.004
9    0.3652
   -0.1402    0.5029    0.1991    0.2189   -0.5870    0.2412   -0.2237   -0.2842   -0.104
1    0.3091


v2 =

  1.0e+003 *

    0.0013         0         0         0         0         0         0         0
0         0
```

```
         0    0.0104         0         0         0         0         0         0         ↙
0        0
         0         0    0.0131         0         0         0         0         0         ↙
0        0
         0         0         0    0.0133         0         0         0         0         ↙
0        0
         0         0         0         0    0.0271         0         0         0         ↙
0        0
         0         0         0         0         0    0.0479         0         0         ↙
0        0
         0         0         0         0         0         0    0.0718         0         ↙
0        0
         0         0         0         0         0         0         0    0.0878         ↙
0        0
         0         0         0         0         0         0         0         0    0.115 ↙
9        0
         0         0         0         0         0         0         0         0         ↙
0   2.3764

>> transpose(v1)*v1
```

$V1^T V1 = I$

```
ans =

    1.0000    0.0000   -0.0000   -0.0000    0.0000    0.0000   -0.0000   -0.0000   -0.000 ↙
0   0.0000
    0.0000    1.0000   -0.0000   -0.0000    0.0000   -0.0000         0    0.0000    0.000 ↙
0   0.0000
   -0.0000   -0.0000    1.0000    0.0000    0.0000    0.0000    0.0000    0.0000   -0.000 ↙
0  -0.0000
   -0.0000   -0.0000    0.0000    1.0000   -0.0000   -0.0000    0.0000   -0.0000    0.000 ↙
0   0.0000
    0.0000    0.0000    0.0000   -0.0000    1.0000   -0.0000   -0.0000    0.0000    0.000 ↙
0  -0.0000
    0.0000   -0.0000    0.0000   -0.0000   -0.0000    1.0000   -0.0000    0.0000    0.000 ↙
0   0.0000
   -0.0000         0    0.0000    0.0000   -0.0000   -0.0000    1.0000    0.0000   -0.000 ↙
0   0.0000
   -0.0000    0.0000    0.0000   -0.0000    0.0000    0.0000    0.0000    1.0000   -0.000 ↙
0   0.0000
   -0.0000    0.0000   -0.0000    0.0000    0.0000    0.0000   -0.0000   -0.0000    1.000 ↙
0  -0.0000
    0.0000    0.0000   -0.0000    0.0000   -0.0000    0.0000    0.0000    0.0000   -0.000 ↙
0   1.0000

>> transpose(v1)*A*v1
```

$V1^T A V1$ is diagonal.

```
ans =

  1.0e+003 *

    0.0013    0.0000    0.0000   -0.0000   -0.0000    0.0000    0.0000    0.0000    0.000 ↙
0   0.0000
    0.0000    0.0104   -0.0000   -0.0000    0.0000   -0.0000    0.0000    0.0000   -0.000 ↙
0  -0.0000
    0.0000    0.0000    0.0131    0.0000   -0.0000    0.0000    0.0000    0.0000    0.000 ↙
0  -0.0000
   -0.0000   -0.0000   -0.0000    0.0133    0.0000    0.0000    0.0000   -0.0000   -0.000 ↙
0  -0.0000
```

6.35

```
   -0.0000   -0.0000   -0.0000    0.0000    0.0271   -0.0000   -0.0000   -0.0000    0.000 ↙
0    0.0000
    0.0000   -0.0000    0.0000   -0.0000   -0.0000    0.0479   -0.0000    0.0000    0.000 ↙
0    0.0000
    0.0000    0.0000    0.0000   -0.0000   -0.0000    0.0000    0.0718   -0.0000   -0.000 ↙
0   -0.0000
   -0.0000    0.0000   -0.0000   -0.0000   -0.0000    0.0000    0.0000    0.0878    0.000 ↙
0    0.0000
   -0.0000    0.0000         0    0.0000   -0.0000   -0.0000   -0.0000   -0.0000    0.115 ↙
9   -0.0000
    0.0000   -0.0000   -0.0000   -0.0000    0.0000    0.0000         0    0.0000   -0.000 ↙
0    2.3764

>> v1*transpose(v1)
```

$V1\ V1^T = I$

```
ans =

    1.0000    0.0000   -0.0000   -0.0000   -0.0000   -0.0000    0.0000    0.0000    0.000 ↙
0    0.0000
    0.0000    1.0000    0.0000   -0.0000   -0.0000    0.0000    0.0000   -0.0000   -0.000 ↙
0   -0.0000
   -0.0000    0.0000    1.0000   -0.0000    0.0000    0.0000    0.0000   -0.0000   -0.000 ↙
0   -0.0000
   -0.0000   -0.0000   -0.0000    1.0000    0.0000    0.0000    0.0000    0.0000    0.000 ↙
0    0.0000
   -0.0000   -0.0000    0.0000    0.0000    1.0000         0   -0.0000    0.0000   -0.000 ↙
0    0.0000
   -0.0000    0.0000    0.0000    0.0000         0    1.0000   -0.0000   -0.0000   -0.000 ↙
0   -0.0000
    0.0000    0.0000    0.0000    0.0000   -0.0000   -0.0000    1.0000   -0.0000    0.000 ↙
0    0.0000
    0.0000   -0.0000   -0.0000    0.0000    0.0000   -0.0000   -0.0000    1.0000    0.000 ↙
0    0.0000
    0.0000   -0.0000   -0.0000    0.0000   -0.0000   -0.0000    0.0000    0.0000    1.000 ↙
0    0.0000
    0.0000   -0.0000   -0.0000    0.0000    0.0000   -0.0000    0.0000    0.0000    0.000 ↙
0    1.0000

>>
```

6.36

Remark:

A matrix (square) M is positive definite if

$$x^T M x > 0$$

for every non-zero x in $\mathbb{R}^n$.

M is assumed $n \times n$.

If $M = A^T A$ it follows that

$A^T T$ is p.d. if

$$x^T A^T A x > 0 \quad \forall x \neq 0$$

$$\Rightarrow \|Ax\|^2 > 0 \quad \forall x \neq 0$$

$$\Rightarrow \|Ax\| \neq 0 \quad \forall x \neq 0$$

$$\Rightarrow Ax \neq 0 \quad \forall x \neq 0$$

$\Rightarrow$ Null space of A must contain only the zero vector.

$\Rightarrow$ nullity of $A = 0$

$\Rightarrow$ rank $A = n$

$\Rightarrow$ A is an invertible matrix i.e. $\det A \neq 0$.

```
Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

To get started, select "MATLAB Help" from the Help menu.
```

Example:

We consider an arbitrary non-singular matrix L. i.e L is invertible rank $L = 3$, $\det L \neq 0$ etc.

```
>> L=[1 3 4;5 6 7;3 2 -1]

L =

     1     3     4
     5     6     7
     3     2    -1

>> A=L*transpose(L)

A =

    26    51     5
    51   110    20
     5    20    14

>> eig(A)

ans =

    0.4029
   12.2118
  137.3853

>> [v1 v2]=eig(A)

v1 =

    0.8209    0.3903    0.4168
   -0.4469   -0.0153    0.8945
    0.3554   -0.9206    0.1619


v2 =

    0.4029         0         0
         0   12.2118         0
         0         0  137.3853

>>
```

$A = L L^T$ is a symmetric positive definite 3×3 matrix.

Eigenvalues of A are all positive.

H. W. 6

(6) We have already argued in the hint that

if the eigenvectors $v_1, \dots, v_n$ are l.d.

it would follow that

either $v_1, \dots, v_{n-1}$ are l.d.

or $v_1, \dots, v_{n-1}$ are l.i and $v_n = 0$

which violates the assumption that $v_n \neq 0$ since it is an eigenvector.

Hence $v_1, \dots, v_{n-1}$ must be l.d. and it would follow that

either $v_1, \dots, v_{n-2}$ are l.d.

or $v_1, \dots, v_{n-2}$ are l.i and $v_{n-1} = 0$

which again violates the assumption that $v_{n-1} \neq 0$ since it is an eigenvector

Proceeding this way we have

v_1, v_2 must be l.d and it would

follow that

either v_1 is l.d. ie $v_1 = 0$

or v_1 is l.i ie $v_1 \neq 0$ and $v_2 = 0$.

since v_1 & v_2 are eigenvectors these cannot be satisfied.

Hence our original hypothesis.

" $v_1, \ldots, v_n$ are l.d"

must be incorrect.

(QED).

(7) $P^{-1}AP = B \Rightarrow AP = PB$.

(a) $\because$ v is an eigenvector of B for an eigenvalue λ we have

$$Bv = \lambda v$$

Hence

$$PBv = P\lambda v = \lambda Pv \quad (\because \lambda \text{ is a scalar})$$

$\Rightarrow APv = \lambda Pv$

$\Rightarrow Pv$ is an eigenvector of A for an eigenvalue λ if $Pv \neq 0$.

Of course $v \neq 0$ & P is nonsingular

$\Rightarrow Pv \neq 0$.

Remark:

Note that $Pv = 0 \Rightarrow$ a non trivial linear combination of the columns of P is zero vector. This is not possible because the columns of P are linearly independent.

Hence $Pv \neq 0$.

— × —.

⑦ ⓑ From Cayley Hamilton Theorem. we have

$$A^3 - c_3 A^2 - c_2 A - c_1 I = 0$$

$$\Rightarrow A^3 = c_3 A^2 + c_2 A + c_1 I.$$

Note that

$$P = [v_1 \ v_2 \ v_3]$$

$$P^{-1} A P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\Rightarrow A [v_1 \ v_2 \ v_3] = [v_1 \ v_2 \ v_3] \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\Rightarrow A v_1 = c_1 v_3$$

$$A v_2 = v_1 + c_2 v_3$$

$$A v_3 = v_2 + c_3 v_3$$

$$\because v_1 = A^2 b - c_3 Ab - c_2 b.$$

$$\Rightarrow A v_1 = A^3 b - c_3 A^2 b - c_2 Ab.$$

$$= (c_3 A^2 b + c_2 Ab + c_1 b) - c_3 A^2 b - c_2 Ab.$$

$$= c_1 b = c_1 v_3.$$ [Verified

$$A v_2 = A^2 b - c_3 A b.$$
$$= (A^2 b - c_3 A b - c_2 b) + c_2 b.$$
$$= v_1 + c_2 v_3 \quad [\text{verified}$$

$$A v_3 = A b = A b - c_3 b + c_3 b.$$
$$= v_2 + c_3 v_3 \quad [\text{verified}.$$

— x —

we know that if λ is an eigenvalue of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}$

with eigenvector $\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$

Pv is an eigenvector of A for eigenvalue λ.

$$Pv = v_1 + \lambda v_2 + \lambda^2 v_3 .$$

(QED)

7(c) We define

$$v_5 = b$$
$$v_4 = Ab - c_5 b$$
$$v_3 = A^2 b - c_5 Ab - c_4 b$$
$$v_2 = A^3 b - c_5 A^2 b - c_4 Ab - c_3 b$$
$$v_1 = A^4 b - c_5 A^3 b - c_4 A^2 b - c_3 Ab - c_2 b$$

The required matrix P would be

$$P = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}.$$

$$\text{Eigenvector} = v_1 + v_2 \lambda_1 + v_3 \lambda_1^2 + v_4 \lambda_1^3 + v_5 \lambda_1^4$$

8

1

writing

$$\Sigma = \Lambda + \mathbb{B}.$$

where

$$\Lambda = \lambda I$$

$$\mathbb{B} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda \mathbb{B} = \mathbb{B} \Lambda = \begin{pmatrix} 0 & \lambda & 0 & & 0 \\ 0 & 0 & \lambda & & \\ 0 & 0 & 0 & \lambda & \\ 0 & 0 & 0 & 0 \cdots & \lambda \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the matrices commute.

$$\text{日}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & & & 0 \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & \text{---} & \text{---} & 1 \\ 0 & 0 & 0 & 0 & \text{---} & & 0 \\ 0 & 0 & 0 & 0 & & \text{---} & 0 \end{pmatrix}$$

$$\text{日}^{16} = \begin{pmatrix} 0 & 0 & \cdot & & 1 \\ & & & & \\ & & 0 & & \\ & & & & \end{pmatrix}$$

Hence $\text{日}^{17} = 0$.

— x —

$$\Sigma^2 = \Lambda^2 + 2\Lambda\text{日} + \text{日}^2$$

$$\Sigma^N = \Lambda^N + {}^N C_1 \Lambda^{N-1}\text{日} + {}^N C_2 \Lambda^{N-2}\text{日}^2 + \cdots$$

$$\cdots {}^N C_{16} \Lambda^{N-16} \text{日}^{16}.$$

Hence

$$^NC_{16}\,\lambda^{N-16}$$

$$\Sigma^N = \begin{pmatrix} \lambda^N & {}^NC_1\lambda^{N-1} & {}^NC_2\lambda^{N-2} & \cdots & \\ & \lambda^N & {}^NC_1\lambda^{N-1} & \cdots & \\ & & \lambda^N & \cdots & \\ & 0 & & \ddots & \\ & & & & \lambda^N \end{pmatrix}$$

② $\boxed{A^N = P\,\Sigma^N P^{-1}}$

$A = P\,\Sigma\,P^{-1}$

$A^2 = P\,\Sigma\,P^{-1}\,P\,\Sigma\,P^{-1}.$

$\quad = P\,\Sigma^2\,P^{-1}$