Home Work 3 (Solutions)

1. $V_1 \times V_2 =$

$$\begin{pmatrix} 3(-9) - 5(-1) \\ (-1)(2) - (1)(-9) \\ 1 \cdot 5 - 3 \cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} -27+5 \\ -2+9 \\ 5-6 \end{pmatrix} = \begin{pmatrix} -22 \\ 7 \\ -1 \end{pmatrix}$$

2. $V_1 \cdot V_2 = 2 + 15 + 9 = 26$

$\|V_1\| = \sqrt{1+9+1} = \sqrt{11}$

$\|V_2\| = \sqrt{4+25+81} = \sqrt{110}$

$$\cos\theta = \frac{26}{\sqrt{1210}} = .7474474$$

$$\theta = 41.63^\circ$$

$\sin\theta = \cdot 664321$

3. $||v_1 \times v_2|| = \sqrt{22^2 + 49 + 1}$

$= \sqrt{484 + 50} = 23 \cdot 10844$

$||v_1|| \, ||v_2|| = \sqrt{1210} = 34 \cdot 785054$

It follows that

$||v_1 \times v_2|| = ||v_1|| \, ||v_2|| \sin\theta = 23 \cdot 10844$

4. $v_1 \times v_2 = (-22 \quad 7 \quad -1)$

$v_3 = (-3 \quad 2 \quad 7)$

$$(v_1 \times v_2) \times v_3 = \begin{pmatrix} 49+2 \\ 3+154 \\ -44+21 \end{pmatrix} = \begin{pmatrix} 51 \\ 157 \\ -23 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 2 & 5 & -9 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -3 & 2 & 7 \end{pmatrix}$$

$$v_2 \times v_3 = \begin{pmatrix} 35 + 18 \\ 27 - 14 \\ 4 + 15 \end{pmatrix} = \begin{pmatrix} 53 \\ 13 \\ 19 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 & 3 & -1 \end{pmatrix}$$

$$v_2 \times v_3 = \begin{pmatrix} 53 & 13 & 19 \end{pmatrix}$$

$$v_1 \times (v_2 \times v_3) = \begin{pmatrix} 57 + 13 \\ -53 - 19 \\ 13 - 159 \end{pmatrix} = \begin{pmatrix} 70 \\ -72 \\ -146 \end{pmatrix}$$

$$(v_1 \times v_2) \times v_3 \neq v_1 \times (v_2 \times v_3)$$

5 $(v_1 \times v_2) \cdot v_3 =$

$66 + 14 - 7 = 73$

$|(v_1 \times v_2) \cdot v_3| = 73$

6 $v_1 \times (v_2 \times v_3) = (70 \quad -72 \quad -146)$

$v_1 \cdot v_3 = -3 + 6 - 7 = 6 - 10 = -4$

$v_1 \cdot v_2 = 2 + 15 + 9 = 26$

$(v_1 \cdot v_3) v_2 = -4 (2 \quad 5 \quad -9)$

$= (-8 \quad -20 \quad 36)$

$(v_1 \cdot v_2) v_3 = 26 (-3 \quad 2 \quad 7)$

$= (-78 \quad 52 \quad 182)$

$(v_1 \cdot v_3) v_2 - (v_1 \cdot v_2) v_3 = (78 - 8, -72, 36 - 182)$

$= (70 \quad -72 \quad -146)$

Likewise

$$(v_1 \times v_2) \times v_3 = (51 \quad 157 \quad -23)$$

$$(v_1 \cdot v_3)\, v_2 = (-8 \quad -20 \quad 36)$$

$$(v_2 \cdot v_3)\, v_1 = (-6+10-63)(1 \quad 3 \quad -1)$$

$$= -59\,(1 \quad 3 \quad -1)$$

$$= (-59 \quad -177 \quad 59)$$

$$(v_1 \cdot v_3)\, v_2 - (v_2 \cdot v_3)\, v_1 =$$

$$(59-8,\ 177-20,\ 36-59)$$

$$= (51 \quad 157 \quad -23)$$

7 $v_1 \times (v_2 \times v_3) = (v_1 \cdot v_3) v_2 - (v_1 \cdot v_2) v_3$

$v_2 \times (v_3 \times v_1) = (v_2 \cdot v_1) v_3 - (v_2 \cdot v_3) v_1 = (v_1 \cdot v_2) v_3 - (v_2 \cdot v_3) v_1$

$v_3 \times (v_1 \times v_2) = (v_3 \cdot v_2) v_1 - (v_3 \cdot v_1) v_2$.

$= (v_2 \cdot v_3) v_1 - (v_1 \cdot v_3) v_2$.

The result follows by summing.

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$$\left.\begin{aligned} \|a \times c\| &= \|a\|\,\|c\| \sin B \\ \|a \times b\| &= \|a\|\,\|b\| \sin C \\ \|b \times c\| &= \|b\|\,\|c\| \sin A \end{aligned}\right\}$$ From definition of cross product.

However

$$\frac{1}{2}\|a \times c\| = \frac{1}{2}\|a \times b\| = \frac{1}{2}\|b \times c\|$$

$$= \text{Area of the triangle.}$$

It follows that

$$\|a \times c\| = \|a \times b\|$$

$$\Rightarrow \|a\|\,\|c\| \sin B = \|a\|\,\|b\| \sin C$$

$$\Rightarrow \frac{\|c\|}{\sin C} = \frac{\|b\|}{\sin B}.$$

Likewise

$$\frac{\|b\|}{\sin B} = \frac{\|a\|}{\sin A}.$$

9 $\boxed{\text{Lagrange Identity simplified}}$

We need the following two identity(ies)

a For a, b, c any three vectors in $\mathbb{R}^3$

$$(a \times b) \cdot c = (b \times c) \cdot a$$

b $$(a \times b) \times c = (a \cdot c) b - (b \cdot c) a$$

Take b and compute for any $d \in \mathbb{R}^3$

$$[(a \times b) \times c] \cdot d =$$

$$[(a \cdot c) b - (b \cdot c) a] \cdot d$$

$$\text{RHS} = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d)$$

Using a we get

$$\text{LHS} = (c \times d) \cdot (a \times b)$$

Thus we have for any four vectors in $\mathbb{R}^3$.

$$(a \times b) \cdot (c \times d) =$$
$$(a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c).$$

Eureka

10 Volume of the parallelopiped formed by vectors a, b, c is

$$a \cdot (b \times c)$$

$\frac{1}{2} \|b \times c\| =$ Area of the base

vertical height $= \|a\| \cos\theta$, where θ is the angle between a and $b \times c$.

Hence $V = \frac{1}{3}\left(\frac{1}{2}\|b \times c\|\right) \cdot \|a\| \cos\theta = \frac{1}{6}\left\|(a \cdot b \times c)\right\|$.