Home Work One Solutions

1. (a) i. Norm of the vector $(1\ \ 2\ \ 1)$ is $\sqrt{1+4+1}=\sqrt{6}$.

   ii. Norm of the vector $(1\ \ 1\ \ -1)$ is $\sqrt{1+1+1}=\sqrt{3}$.

   iii. Norm of the vector $(2\ \ 3\ \ 0)$ is $\sqrt{4+9+0}=\sqrt{13}$.

   (b) The cosine of the angle θ between any two vectors a and b is given as follows

$$cos\ \theta = \frac{a.b}{\|a\|\|b\|}.$$

   Angle between the vectors $(1\ \ 2\ \ 1)$ and $(1\ \ 1\ \ -1)$ is

$$cos^{-1}\frac{1+2-1}{\sqrt{6}\sqrt{3}} = \frac{2}{\sqrt{18}} = 61.87 \text{ degrees} = 1.0799 \text{ radians}$$

   Angle between the vectors $(1\ \ 2\ \ 1)$ and $(2\ \ 3\ \ 0)$ is

$$cos^{-1}\frac{2+6+0}{\sqrt{6}\sqrt{13}} = \frac{8}{\sqrt{78}} = 25.07 \text{ degrees} = .437 \text{ radians}$$

   Angle between the vectors $(1\ \ 1\ \ -1)$ and $(2\ \ 3\ \ 0)$ is

$$cos^{-1}\frac{2+3+0}{\sqrt{3}\sqrt{13}} = \frac{5}{\sqrt{39}} = 36.81 \text{ degrees} = .642 \text{ radians}$$

   (c) Let u be the vector $(3\ \ 4\ \ -5)$. We have $\|u\| = \sqrt{3^2+4^2+(-5)^2} = \sqrt{50}$. The required two vectors are $\frac{u}{\|u\|}$ and $-\frac{u}{\|u\|}$ given by

$$\pm(\frac{3}{\sqrt{50}}\ \ \frac{4}{\sqrt{50}}\ \ \frac{-5}{\sqrt{50}})$$

   (d) Let a and b be the two vectors $(1\ \ 3\ \ 9)$ and $(2\ \ 4\ \ -6)$ respectively.

$$\|a\| = \sqrt{91}, \|b\| = \sqrt{56}, \|a+b\| = \sqrt{67}$$

   Moreover

$$|a.b| = 40 \text{ and } \|a-b\| = \sqrt{227}$$

   Clearly $\sqrt{67} < \sqrt{91}+\sqrt{56}$, $40 < \sqrt{91}\ \sqrt{56}$ and $\sqrt{227} > \sqrt{91}-\sqrt{56}$

2. (a)

$$\|(1\ \ 1)\| = \sqrt{1^2+1^2} = \sqrt{2}$$

   Angle between $(1\ \ 1)$ and $(1\ \ 0)$ is

$$cos^{-1}\frac{1}{\sqrt{2}} = 45 \text{ degrees}$$

   (b)

$$\|(2\ \ -3)\| = \sqrt{2^2+(-3)^2} = \sqrt{13}$$

   Angle between $(2\ \ -3)$ and $(1\ \ 0)$ is

$$cos^{-1}\frac{2}{\sqrt{13}} = -56.3 \text{ degrees}$$

   (c)

$$\|(\sqrt{3}\ \ 1)\| = \sqrt{3+1^2} = \sqrt{4} = 2$$

   Angle between $(\sqrt{3}\ \ 1)$ and $(1\ \ 0)$ is

$$cos^{-1}\frac{\sqrt{3}}{2} = 30 \text{ degrees}$$

(d)

$$\|(5 \;\; 12)\| = \sqrt{5^2+12^2} = \sqrt{169} = 13$$

Angle between (5 12) and (1 0) is

$$cos^{-1}\frac{5}{13} = 67.38 \text{ degrees}$$

(e)

$$\|(-5 \;\; -12)\| = \sqrt{(-5)^2+(-12)^2} = \sqrt{169} = 13$$

Angle between $(-5 \;\; -12)$ and (1 0) is

$$cos^{-1}\frac{-5}{13} = -112.62 \text{ degrees}$$

3. (a) If the vector is $(a \;\; b)$, it follows that $a^2+b^2=1$, $\frac{b}{a} = \frac{1}{\sqrt{3}}$. Solving, we get $a = \frac{\sqrt{3}}{2}$ and $b = \frac{1}{2}$. The required unit vector is $(\frac{\sqrt{3}}{2} \;\; \frac{1}{2})$.
   (b) The required vector is an unit vector making an angle of -30 degrees with respect to the $x - axis$. The vector is $(.866 \;\; -.5)$
   (c) The vector is $(\frac{3}{5} \;\; -\frac{4}{5})$.
   (d) The slope of the tangent vector is 4. A unit vector with slope 4 is given by $\pm(\frac{1}{\sqrt{17}} \;\; \frac{4}{\sqrt{17}})$
   (e) The required unit normal vector is given by $(\frac{-4}{\sqrt{17}} \;\; \frac{1}{\sqrt{17}})$

4. Assume that the cube is of length, breadth and height given by a. It can be aligned in such a way that the diagonal is the vector $d = (a \;\; a \;\; a)$, the face diagonal is the vector $f = (a \;\; a \;\; 0)$ and the edge is the vector $e = (a \;\; 0 \;\; 0)$.

   (a) Angle between d and e is given by $\cos^{-1}\frac{a^2}{a\sqrt{3}\times a} = 54.74$ degrees.
   (b) Angle between d and f is given by $\cos^{-1}\frac{2a^2}{a\sqrt{3}\times a\sqrt{2}} = 35.26$ degrees.
   (c) If $a \neq 0$ and $b \neq 0$, then the line $ax+by+c=0$ can be parameterized in vector as

$$\{u + tv : t \text{ is any real number}\}$$

   and where $u = (-\frac{c}{a} \;\; 0)$ and $v = (\frac{c}{a} \;\; -\frac{c}{b})$. The vector $(a \;\; b)$ is clearly perpendicular to the vector v. If $a = 0$, the line is given by $by + c = 0$ which is parallel to the x axis. The vector $(0 \;\; b)$ is clearly parallel to the x axis. If $b = 0$, the line is given by $ax + c = 0$ which is parallel to the y axis. The vector $(a \;\; 0)$ is clearly parallel to the y axis.

5. The vectors A and B are defined as follows:

$$A = (2 \quad 5), \;\; B = (3 \;\; -4)$$

. We have $\|A\| = \sqrt{29}$, $\|B\| = 5$ and $A.B = 6 - 20 = -14$. It follows that

(a)

$$\text{proj}_A B = \frac{B.A}{\|A\|^2} A = -\frac{14}{29}(2 \quad 5)$$

(b)

$$\text{proj}_B A = \frac{A.B}{\|B\|^2} B = -\frac{14}{25}(3 \quad -4)$$

(c)

$$C = A - \text{proj}_B A = \frac{1}{25}(92 \quad 69)$$

(d)

$$C.B = \frac{1}{25}(92 \quad 69).(3 \;\; -4) = \frac{1}{25} \times (276 - 276) = 0.$$

Hence the vectors C and B are orthogonal to each other.