H. W. 7 (Hint)

① (a) $\Lambda = \begin{pmatrix} 5 & & & 0 \\ & 7 & & \\ & & 13 & \\ 0 & & & 17 \end{pmatrix}$

$P = [v_1 \ \ v_2 \ \ v_3 \ \ v_4]$

Define $A = P^{-1} \Lambda P$

(b) Let u_1, u_2, u_3, u_4 be the required orthonormal vectors.

(c) $Q = [u_1 \ \ u_2 \ \ u_3 \ \ u_4]$

We know that Q is an orthogonal matrix.

$Q^{-1} = Q^T$

Define $B = Q^T \Lambda Q$.

(d) writing

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

we have

$$x^T B x = 100$$

$$\Rightarrow x^T Q^T \Lambda Q x = 100$$

Define $y = Qx$, we write

$$y^T \Lambda y = 100$$

If $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ we conclude

$$5y_1^2 + 7y_2^2 + 13y_3^2 + 17y_4^2 = 100$$

(e) In the x co-ordinate the line l is the span of the vector

$$\begin{pmatrix} 3 \\ 4 \\ -5 \\ -1 \end{pmatrix}.$$

In the y-coordinate the same line l is the span of the vector

$$Q \begin{pmatrix} 3 \\ 4 \\ -5 \\ -1 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \text{ say.}$$

writing $\left.\begin{aligned} Y_1 &= \xi_1 t \\ Y_2 &= \xi_2 t \\ Y_3 &= \xi_3 t \\ Y_4 &= \xi_4 t \end{aligned}\right\}$ we get $t^2 = \dfrac{100}{5\xi_1^2 + 7\xi_2^2 + 13\xi_3^2 + 17\xi_4^2}$

$$t = \frac{\pm 10}{\sqrt{5\xi_1^2 + 7\xi_2^2 + 13\xi_3^2 + 17\xi_4^2}}$$

Two points of intersection are

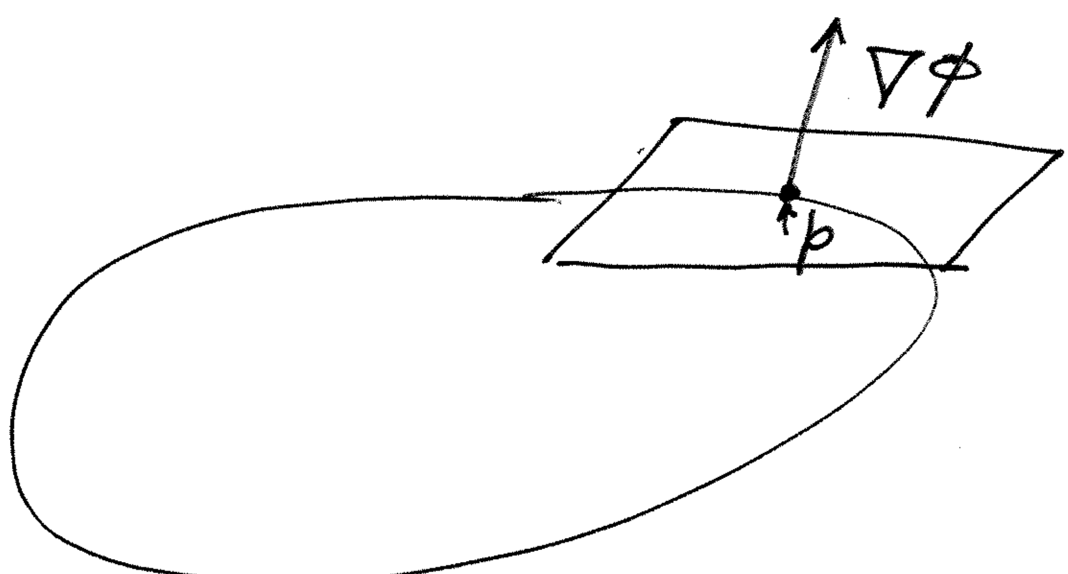
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \pm \begin{pmatrix} \frac{10\xi_1}{\Delta} \\ \frac{10\xi_2}{\Delta} \\ \frac{10\xi_3}{\Delta} \\ \frac{10\xi_4}{\Delta} \end{pmatrix}$$

where

$$\Delta = \sqrt{5\xi_1^2 + 7\xi_2^2 + 13\xi_3^2 + 17\xi_4^2}$$

(f) Define

$$\Phi(y_1, y_2, y_3, y_4) = 5y_1^2 + 7y_2^2 + 13y_3^2 + 17y_4^2$$



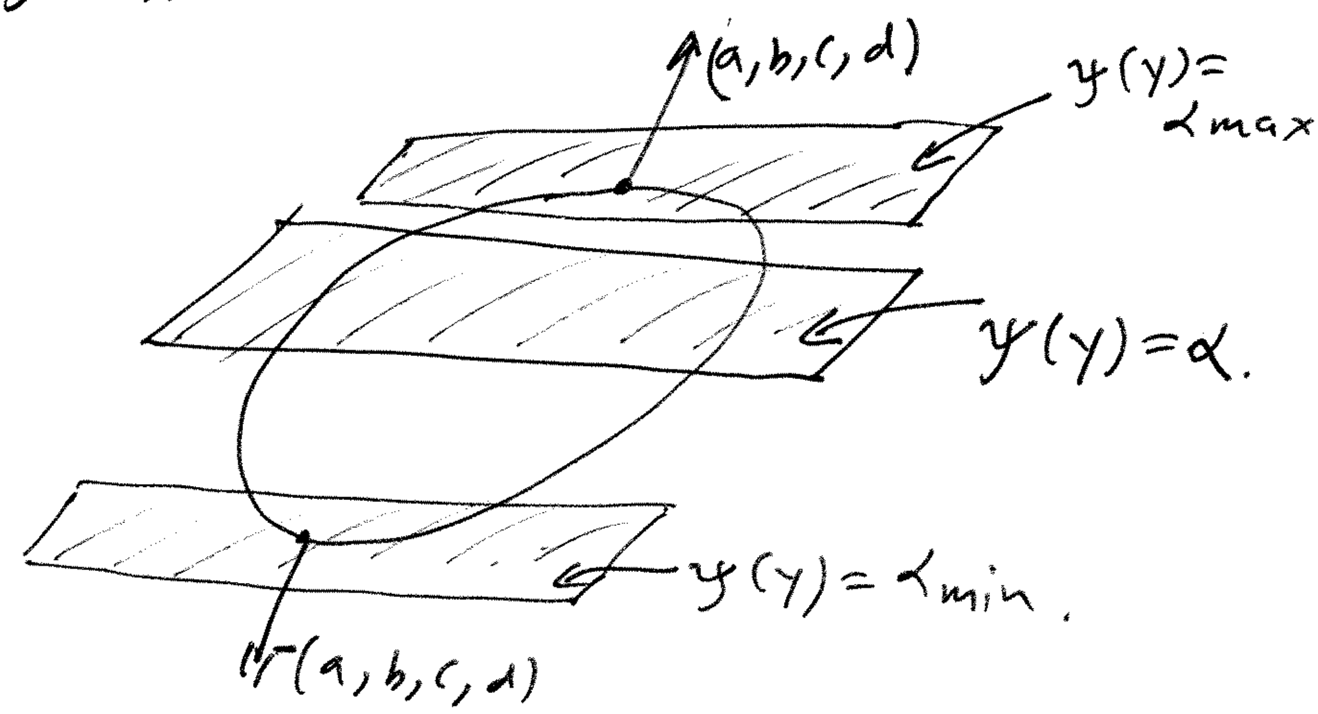
$$\nabla\phi = \left(\frac{\partial\Phi}{\partial y_1} \quad \frac{\partial\Phi}{\partial y_2} \quad \frac{\partial\Phi}{\partial y_3} \quad \frac{\partial\Phi}{\partial y_4}\right)$$

is a vector perpendicular to the tangent plane at p

- Write down the equation of the plane in the y co-ordinate.
- Rewrite the same equation in the x-co-ordinate.

(g) Transform the function $\phi(x_1, x_2, x_3, x_4)$ to the f^n

$$\psi(y_1, y_2, y_3, y_4) = a y_1 + b y_2 + c y_3 + d y_4.$$



Gradient of the ellipsoid is.

$$(10y_1, 14y_2, 26y_3, 34y_4)$$

The max$^{\underline{m}}$ and min$^{\underline{m}}$ value of ψ are pts where the plane

$$ay_1 + by_2 + cy_3 + dy_4 = \alpha$$

is tangent to the ellipsoid.

ie at

$$y_1 = \pm a/10, \quad y_2 = \pm b/14, \quad y_3 = \pm c/26, \quad y_4 = \pm d/34.$$

Value of ψ at that point

$$\psi(\cdot) = \pm\left(\frac{a^2}{10} + \frac{b^2}{14} + \frac{c^2}{26} + \frac{d^2}{34}\right)$$

(2)(9)

The matrix A must have eigenvalues at $\pm 2i, \pm 4i$.

$$A = \begin{pmatrix} 0 & 2 & & \\ -2 & 0 & & \\ & & 0 & 4 \\ & & -4 & 0 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} \cos 2t & \sin 2t & & \\ -\sin 2t & \cos 2t & & \\ & & \cos 4t & \sin 4t \\ & & -\sin 4t & \cos 4t \end{pmatrix}$$

The trick is to find the initial condition.

(b) The matrix A has to have a repeated eigenvalue at:

$$3 \pm 7i, \ 3 \pm 7i$$

$$A = \begin{pmatrix} 3 & 7 & 1 & 0 \\ -7 & 3 & 0 & 1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & -7 & 3 \end{pmatrix}$$

7 Define

$$y_0 = 1, \; y_1 = 1, \; y_2 = 2, \; y_3 = 3, \; \cdots$$

We have

$$y_k = y_{k-1} + y_{k-2} \qquad k = 2, 3, 4, \cdots$$

Let $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $x_{k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} x_k$

We obtain.

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; x_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \; \cdots$$

In general

$$x_j = \begin{pmatrix} y_j \\ y_{j+1} \end{pmatrix} \quad \text{i.e} \quad \boxed{y_j = (1 \;\; 0)\, x_j}$$

$$y_{100} = \begin{pmatrix} 1 & 0 \end{pmatrix} X_{100}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} A^{100} X_0$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Calculate $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{100}$.

(8)

Let

$$S = I + A + A^2 + \cdots + A^{100}$$

$$SA = \quad A + A^2 + \cdots + A^{100} + A^{101}$$

$$S - SA = I - A^{101}$$

$$\Rightarrow S(I - A) = (I - A^{101})$$

$$\Rightarrow S = (I - A^{101})(I - A)^{-1}$$

— x —

Show that if

$$S_N = I + A + \cdots + A^N.$$

then

$$S_N = (I - A^{N+1})(I - A)^{-1}$$

If the eigenvalues of A have magnitude < 1 it follows that

$$\underset{N \to \infty}{Lt} A^N = 0 .$$

$$\underset{N \to \infty}{Lt} S_N = (I - A)^{-1}.$$

Problem 4

① $\dot{x}_1 = -6x_1 - x_2$

$\dot{x}_2 = -9x_1 - 6x_2$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -6 & -1 \\ -9 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$-3 \longleftrightarrow \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$\lambda_1 = -3 \qquad \lambda_2 = -9$

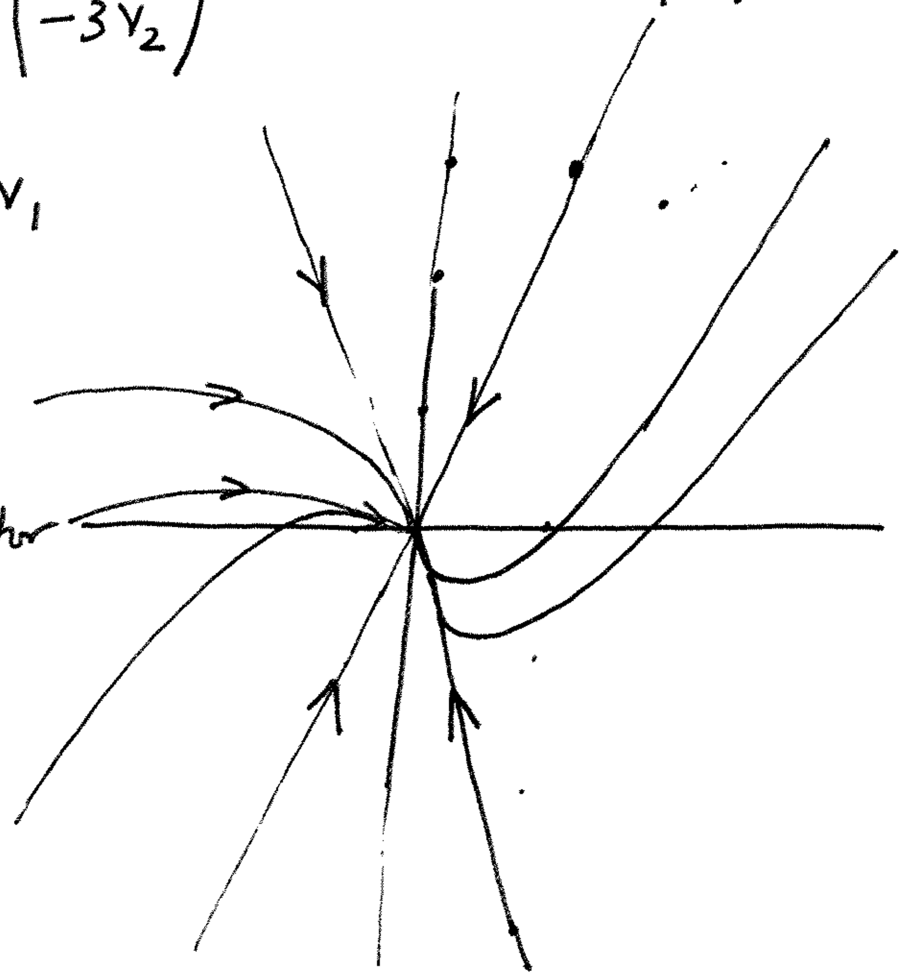
$$-9 \longleftrightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -1 \\ -9 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -3v_1 \\ -3v_2 \end{pmatrix}$$

$-6v_1 - v_2 = -3v_1$

$\boxed{v_2 = -3v_1}$

$\begin{pmatrix} 1 \\ -3 \end{pmatrix} \leftarrow$ eigenvector



2

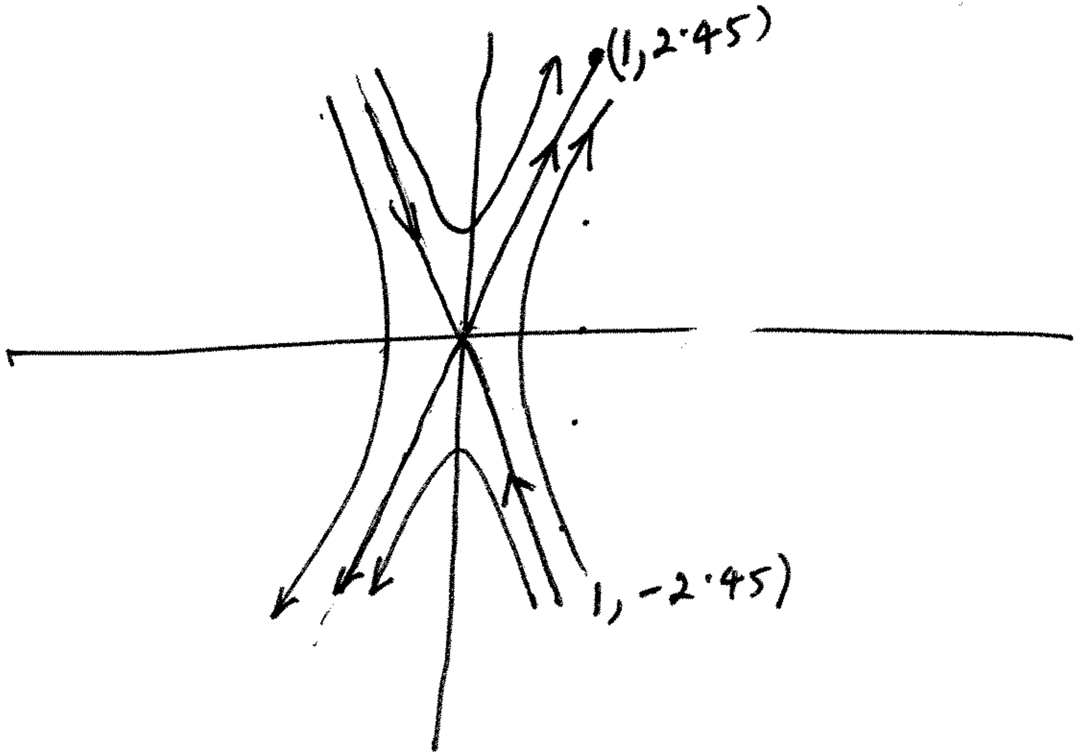
$\dot{x}_1 = 2x_1 + x_2$

$\dot{x}_2 = 6x_1 + 2x_2$

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 2 \end{pmatrix}$$

$\lambda_1 = 4 \cdot 45$ $\lambda_2 = - \cdot 45$

$$v_1 = \begin{pmatrix} 1 \\ 2 \cdot 45 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -2 \cdot 45 \end{pmatrix}$$



③ $\dot{x}_1 = -x_1$

$\dot{x}_2 = -5x_1 - x_2$.

$$A = \begin{pmatrix} -1 & 0 \\ -5 & -1 \end{pmatrix}$$

Eigen value $\lambda = -1$ repeated twice

$v_1 = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$ eigenvector.

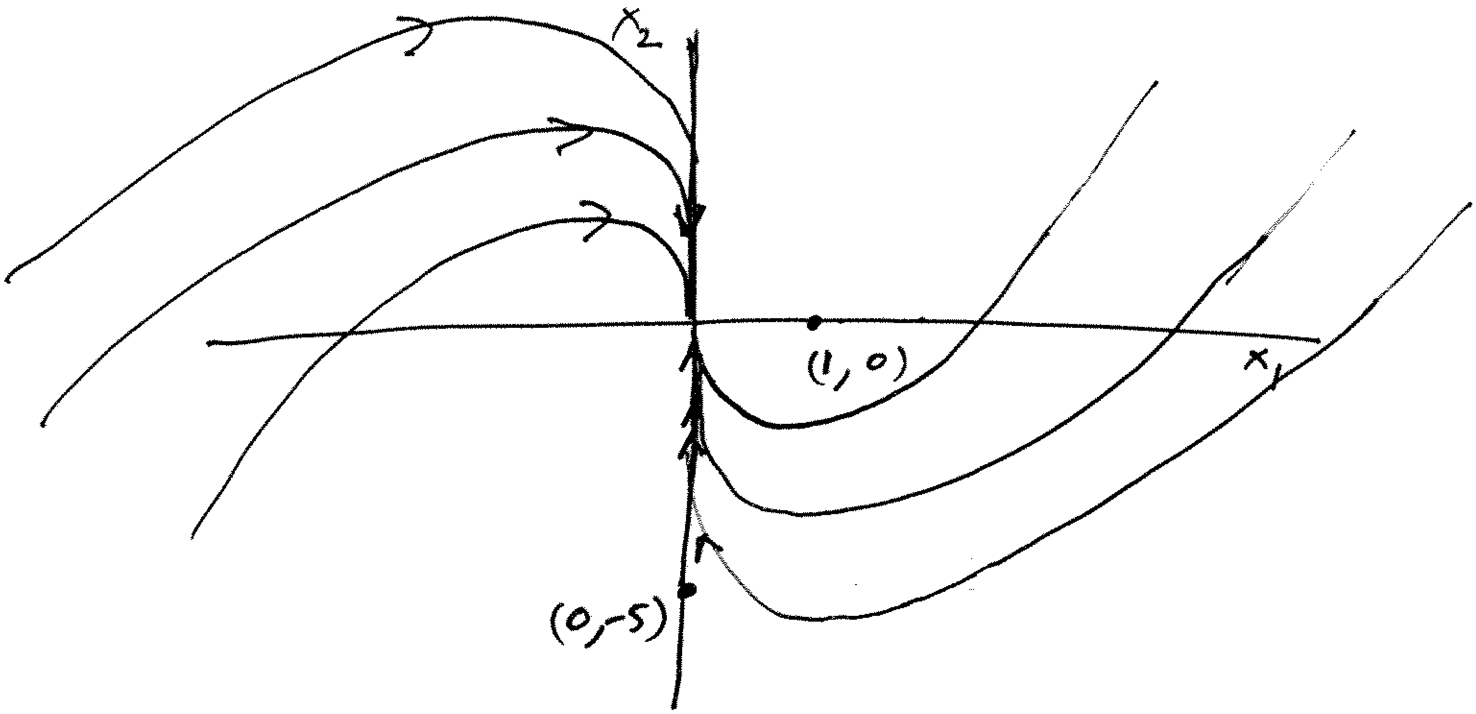
$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gen eigenvector.

$$Av_1 = \lambda v_1$$

$$Av_2 = \lambda v_2 + v_1$$

$$e^{At} v_1 = e^{\lambda t} v_1 = e^{-t} \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$e^{At} v_2 = e^{\lambda t} v_2 + t e^{\lambda t} v_1 = \begin{pmatrix} e^{-t} \\ -5te^{-t} \end{pmatrix}$$



$x_2(t) = -5te^{-t}$

$x_1(t) = e^{-t}$.

$\text{slope} = \dfrac{x_2}{x_1} = -5t$

slope $\to \infty$ as $t \to \infty$.

4

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -9x_1$$

$$A = \begin{pmatrix} 0 & 1 \\ -9 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \qquad P^{-1}AP = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

$$P\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -y_2 \\ 3y_1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

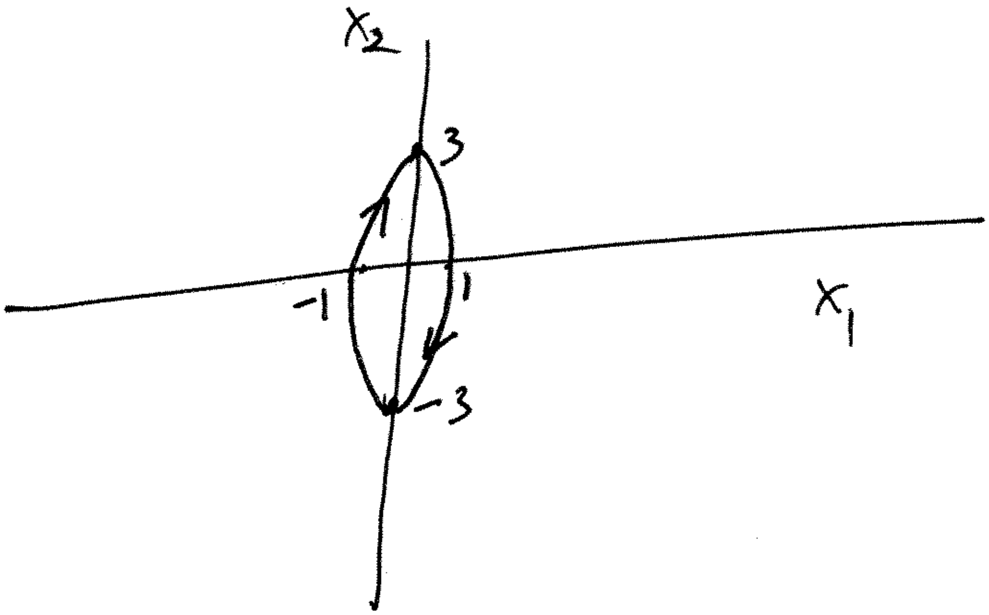
$$y_1^2 + y_2^2 = 1.$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$$

$$y_2 = -x_1.$$
$$y_1 = \frac{1}{3}x_2.$$

$$\frac{1}{9}x_2^2 + x_1^2 = 1.$$

$$\boxed{x_2^2 + 9x_1^2 = 9}.$$



3 (a)

$$\boxed{\begin{gathered}\ddot{x} + \dot{x} + x = \sin 4t \\ x(0) = 5 \quad \dot{x}(0) = 15\end{gathered}}$$

Homogeneous sol$^{\underline{n}}$:

$$\ddot{x} + \dot{x} + x = 0$$

char pol. $\lambda^2 + \lambda + 1 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$= \sigma \pm i\omega \qquad \boxed{\sigma = -\frac{1}{2}, \omega = \frac{\sqrt{3}}{2}}$$

$$x_h(t) = a e^{-t/2} \sin \frac{\sqrt{3}}{2} t + b e^{-t/2} \cos \frac{\sqrt{3}}{2} t.$$

$$x_p(t) = c \sin 4t + d \cos 4t.$$

To find c, d we write

$$\dot{x}_p(t) = 4c \cos 4t - 4d \sin 4t.$$

$$\ddot{x}_p(t) = -16c \sin 4t - 16d \cos 4t.$$

$$\ddot{x}_p + \dot{x}_p + x_p = (c - 4d - 16c) \sin 4t + (d + 4c - 16d) \cos 4t = \sin 4t$$

$$\Rightarrow \boxed{-4d - 15c = 1;\ 4c - 15d = 0}$$

$$\begin{pmatrix} -15 & -4 \\ 4 & -15 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$c = \frac{-15}{225+16} = \frac{-15}{241}$$

$$d = \frac{-4}{241}.$$

$$x(t) = x_h(t) + x_p(t)$$

$$= a e^{-t/2} \sin \frac{\sqrt{3}}{2} t + b e^{-t/2} \cos \frac{\sqrt{3}}{2} t.$$

$$- \frac{15}{241} \sin 4t - \frac{4}{241} \cos 4t.$$

$$x(0) = b - \frac{4}{241} = 5 \qquad \boxed{b = 5 + \frac{4}{241}.}$$

Now find 'a' from $\dot{x}(0) = 15$.