$\boxed{\text{Hint for H.W. 6}}$

① ⓓ

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \underline{0}$$

This is a nilpotent matrix.

① ⓔ

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \phi & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is not nilpotent

I would use matlab to compute its exponential.

(2) (a)

If A has eigenvalues at $\lambda_1, \lambda_2, \lambda_3$

A^2 " " at $\lambda_1^2, \lambda_2^2, \lambda_3^2$

(3) (a) The matrix M has eigenvalue 12 repeating 4 times.

choose $\lambda = 12$

calculate $M - \lambda I$; we get 0 matrix.

If $M v_1 = \lambda v_1$

$M v_2 = \lambda v_2 + v_1$

we get $(M - \lambda I) v_1 = 0$ $\Rightarrow (M - \lambda I)^2 v_2 = 0$

$(M - \lambda I) v_2 = v_1$

Choose v_2 : $(M - \lambda I) v_2 \neq 0$, $(M - \lambda I)^2 v_2 = 0$

One choice is $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $v_1 = (M - \lambda I) v_2 = \begin{pmatrix} 33 \\ -60 \\ 45 \\ -12 \end{pmatrix}$

$$v_1 = \begin{pmatrix} 33 \\ -60 \\ 45 \\ -12 \end{pmatrix} ; \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ is a chain of gen. eigenvectors.

$$v_1 = (M - 12I)\, v_2$$

$$(M - 12I)^2\, v_2 = 0$$

Likewise

$$v_3 = \begin{pmatrix} 315 \\ -756 \\ 639 \\ -180 \end{pmatrix} ; \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$ is another chain of gen. eigenvectors.

— x —

B.

(5) In this problem please do not manufacture a diagonal matrix. I am looking for a genuine (i.e. non diagonal) matrix. One way to do this is the following

- Let A be any symmetric matrix. (Dont choose A to be diagonal).
- Eigenvalues of A are real, but could be negative. Write

  $$B = \exp(A)$$ ← use matlab here.
- If λ is an eigenvalue of A

  e^{λ} is an " " B

  Hence eigenvalues of B are all positive.
- $B^T = (e^A)^T = e^{A^T} = e^A = B$

  Hence is also symmetric.

B is the required answer.

- Eigenvectors of B are all l.i. and orthogonal to each other (sp. property of symmetric p.d. matrices)
- Let P be the matrix of eigenvectors of B. Then
  $$P^T = P^{-1} \cdot \text{ because } \underbrace{PP^T = I}_{\text{property of orthogonal eigenvectors}}.$$
- $P^{-1}BP = P^TBP$ is a diagonal matrix of eigenvalues of B.